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On a coupled Stefan-like problem in thermo-visco-plastic rheology[1,2]

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Abstract

This paper is a study of the existence of solutions of Stefan-like problems describing solidification and melting of rock and metallic alloys. The governing equations are derived and the model problem is given. The main result represents the existence theorem. The proof proceeds by a regularization of the nonlinear degenerated terms and of the nondifferentiable functional $j(.)$. Sequences of regularized solutions are obtained by the Galerkin approximations. For proving the solution monotonicity arguments are used. The problem investigated represents the model problem of solidification and melting as well as simulation of geodynamic processes in the Earth and simulation of critical situations (e.g. explosions of HLRWDSs) in regions, where high-level radioactive waste disposal systems (HLRWDSs) will be situated.

Keywords: Thermoplasticity; Stefan problem; Solidification; Melting, Variational equality and inequality; FEM; Geodynamics; Technology

AMS classification: 35K55; 35K60; 76A99; 76T05; 73N05; 80A20

1. Introduction

Many physical processes connected with heat flow and diffusion involving phase-change phenomena give rise to free boundary problems for parabolic partial differential equations of the Stefan type. Historically, the first paper was given by Joseph Stefan (1835–1893) [35] and was concerned with the melting of ice at 0°C. A concept of weak solution of the Stefan problem was introduced by Kamenomotskaya [16, 17] and was then analyzed by means of smoothing techniques as developed in [10, 19, 31] and of monotonicity methods developed by Brezis [2] and Lions [21]. A very

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rich literature exists for numerical solutions of Stefan problems. The problem is nonlinear so that numerically the problem was solved by finite difference and finite element methods, e.g. [9, 30, 5, 13, 14, 8, 27, 28, 34]. and in variational inequality approach in [6, 32], etc. The algorithms are then based on the nonlinear SOR method, developed e.g. in [38, 39, 8] or on nonlinear conjugate gradient method. The coupled contact-two-phase Stefan-like problem is solved in [24–26].

The interest of geophysical and technological disciplines for numerical modelling has encouraged the development of corresponding numerical methods. Not only in geodynamics but also in technological practice, processes connecting with heating (melting and recrystallization) and freezing (solidification) play an important role instead of processes connecting with moving masses. As a first step, simulations related to this topic consist in studying the macroscopic heat transfer mechanism, modelled by the so-called two-phase Stefan problem. Since problems connecting with melting, recrystallization as well as solidification play an important role in geodynamic processes in the core, mantle and as well as in the lithosphere of the Earth, investigations of corresponding high performance numerical methods have a great importance for simulation and further understanding of the geodynamic processes inside the Earth. At present it is applied to the modelling and simulation geomechanic and geodynamic processes in regions where the high level radioactive waste repositories will be built. In "classical" problems of technology the two-phase Stefan problems with convection in the fluid phase are investigated [1, 4]. Similar problems are applied also in geodynamics [33]. In both problems the rheology is taken as Newtonian rheology oftentimes in the Bussinesq's approximation [1, 12, 33]. This approximation depends on Rayleigh's, Prandtl's, etc. numbers, which change in relatively great value intervals. Therefore, the main goal of this paper is to give an optimal method, i.e., optimal from the better physical and rheological approximation point of view as well as from the high performance computation possibility point of view. The Bingham rheology is an optimal rheology as for the case if the threshold of plasticity $\hat{g}$ is equal to zero, then we have the usual present case of Newtonian rheology and if the threshold of plasticity $\hat{g} \to \infty$ then the medium is absolutely rigid and between them we can model all types of visco-plastic materials. It is evident that both results can be compared because the threshold of plasticity is determined by the Mises relation, hence by the velocities of seismic P and S waves and the density (Lamé coefficients and density). Therefore the approximation of the rheology inside the Earth is close to reality. Moreover, together with two-phase Stefan problem the approximation of the reality is much more realistic. It is evident that similar considerations are also valid for the technological problems.

It is evident that due to the extremely complicated shape of the front for melting, recrystallization or solidification, respectively, cannot be described in full. As a result we practically obtain the phase change zones only, as the forecast of the fine geometry would need a very expensive computation as well as the necessity of regularity of the front, where both phases are parallelly at the same time (the so-called mushy zone). To study these problems we have proposed to average variables over the phases [37] so that instead of the fine geometry, a smoother phase change zone is investigated. The principle of this average is based on the integration of the variables over an elementary domain, where each variable is in its own phase. The effect of these averaging processes are to define zones of intermediate state (mushy zones), in between both phases. Within these mushy zones, the relative proportion of each phase is given by the volumic fractions f_S, f_L, where $f_S + f_L = 1$. In the mushy zones both phases are microscopically parallelly present. From the mathematical point of view the mushy zones mean the regions where the operator of heat equation is degenerated. In the phase diagram, this corresponds to the phase changes either at null concentration and temperature of melting (fusion), or at concentration c from the interval, corresponding to the concentration for which the solid rock begins to melt and the concentration for which the liquid rock begins to solidify and eutectic temperature, i.e., temperature, under which no liquid remains, whatever the concentration. The latent heat represents the energy necessary for a complete change of phase of a unit volume at a temperature of fusion (melting) and null concentration.

2. Physical model

The physical model is derived from the conservation principle. The conservation of momentum in its differential form gives the equation of motion, the mass conservation specifies the rocks as the incompressible rocks and gives also the solute equation. The energy conservation then gives the heat equation. The 'classical' approach in the phase change problems of the first order (melting, recrystallization, solidification) is based upon Boussinesq's hypothesis assuming that the density variation is neglected except in the force term. Then the obtained equation is valid in the liquid phase only when the solid phase is assumed to be static. These assumptions lead to the Navier–Stokes equations. But in this paper for better approximation of the rheology we shall assume that the rheology is of the visco-plastic Bingham's type.

In the case of thermo-visco-plastic Bingham's rheology from the conservation of momentum we obtain the equations of motion

$$\rho\left(\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}\right)=\frac{\partial \tau_{ij}}{\partial x_j}+\rho f_i \quad \text{in } \Omega\times I,\ i,j=1,2(3),\ I=(t_0,t_1), \tag{1}$$

where u_i are components of flow velocity vector, ρf_i components of body forces.

The stress–strain rate relation (the constituent law) can be derived from the dissipation function $\tau_{ij}D_{ij}$, where τ_{ij} is the stress tensor and D_{ij} is the strain-rate tensor defined as

$$D_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\quad i,j=1,2(3), \tag{2}$$

assuming that the dissipation function depends only on the strain rate tensor, and where $\boldsymbol{u}=(u_i)$ is the flow velocity vector. Then we can write

$$\tau_{ij}D_{ij}=\mathscr{D}_1(\boldsymbol{D})+\mathscr{D}_2(\boldsymbol{D}),$$

where $\mathscr{D}_1$ and $\mathscr{D}_2$ are positive homogeneous functions of order 1 and 2, respectively, in the components of the strain rate tensor $\boldsymbol{D}$. Let us put

$$\mathscr{D}_1=\frac{\partial\mathscr{D}_1}{\partial D_{kl}}D_{kl},\qquad \mathscr{D}_2=\frac{1}{2}\frac{\partial\mathscr{D}_2}{\partial D_{kl}}D_{kl}. \tag{3}$$

Since, we shall assume that the rocks are incompressible, then

$$\operatorname{div}\boldsymbol{v}=0,\quad \text{i.e.,}\quad D_{kk}=0, \tag{4}$$

which follows from the mass conservation law. Hence, (3) and (2) we find

$$\tau_{ij}=-p\delta_{ij}+\frac{\partial\mathscr{D}_1}{\partial D_{ij}}+\frac{1}{2}\frac{\partial\mathscr{D}_2}{\partial D_{ij}}, \tag{5}$$

where p is a scalar, independent of the strain rate tensor D_{ij} and represents the spherical part of the stress tensor and has a meaning of the pressure and δ_{ij} is the Kronecker symbol.

In the next, we shall assume that rocks are assumed to be isotropic, then the scalars $\mathscr{D}_1$ and $\mathscr{D}_2$ are functions of the invariants of the strain rate tensor D_{ij} only. For the case of the Bingham's rheology

$$\mathscr{D}_1 = 2\hat{g}D_{\mathrm{II}}^{1/2}, \qquad \mathscr{D}_2 = 4\hat{\mu}D_{\mathrm{II}}, \qquad D_{\mathrm{II}} = \tfrac{1}{2}D_{ij}D_{ij}, \tag{6}$$

where D_{II} is invariant of the strain rate tensor D_{ij}, and $\hat{g}$, $\hat{\mu}$ represent the thresholds of plasticity and viscosity. Due to [29] the thermal stress τ_{ij}^{T} satisfies

$$\tau_{ij}^{\mathrm{T}} = \beta_{ij}(T - T_0),$$

where β_{ij} is a coefficient of thermal expansion, $T(\boldsymbol{x},t)$, $T_0(\boldsymbol{x})$, $T_0(\boldsymbol{x}) > 0$, are a temperature and an initial temperature in which the medium is in the initial stress–strain state. Thus the constituent law is as follows

$$\tau_{ij} = -p\delta_{ij} + \hat{g}D_{ij}D_{\mathrm{II}}^{-1/2} + 2\hat{\mu}D_{ij} + \beta_{ij}(T - T_0), \tag{7}$$

having sense for $D_{\mathrm{II}} \neq 0$ only. To derive the inversion stress–strain rate relations for the Bingham's part of the stress tensor we first define the invariant of the stress tensor as

$$\tau_{\mathrm{II}} = \tfrac{1}{2}\tau_{ij}^{\mathrm{D}}\tau_{ij}^{\mathrm{D}}, \tag{8}$$

where $\tau_{ij}^{\mathrm{D}} = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij}$ is the deviator of the stress tensor. Then with (7) we find

$$\tau_{\mathrm{II}} = (\hat{g} + 2\hat{\mu}D_{\mathrm{II}}^{1/2})^2. \tag{9}$$

Hence,

$$\tau_{\mathrm{II}}^{1/2} \geqslant \hat{g}, \tag{10}$$

and thus

$$D_{ij} = (2\hat{\mu})^{-1}(1 - \hat{g}\tau_{\mathrm{II}}^{-1/2})\tau_{ij}^{\mathrm{D}}. \tag{11}$$

Hence and (6c)

$$D_{\mathrm{II}} = \tfrac{1}{2}D_{ij}D_{ij} = (2\hat{\mu})^{-2}(\tau_{\mathrm{II}}^{1/2} - \hat{g})^2.$$

Then

$$D_{\mathrm{II}} = 0 \text{ for } \tau_{\mathrm{II}}^{1/2} - \hat{g} \leqslant 0, \text{ i.e., } \quad \text{for } \tau_{\mathrm{II}}^{1/2} \leqslant \hat{g}$$

and

$$D_{\mathrm{II}} \neq 0 \text{ for } \tau_{\mathrm{II}}^{1/2} - \hat{g} > 0, \text{ i.e., } \quad \text{for } \tau_{\mathrm{II}}^{1/2} > \hat{g}.$$

Then the inverse constituent law in the thermo-Bingham rheology can be written in the form

$$\tau_{II}^{1/2} \leqslant \hat{g} \Leftrightarrow D_{ij} = 0,$$
$$\tau_{II}^{1/2} > \hat{g} \Leftrightarrow D_{ij} = (2\hat{\mu})^{-1}(1 - \hat{g}\tau_{II}^{-1/2})\tau_{ij}^{D} + \gamma_{ij}(T - T_0). \tag{12}$$

If $\hat{g} = 0$, then (7) represents the constituent law for the Newtonian fluid, i.e., for the well-known viscous incompressible fluid. For $\hat{g} \to \infty$ we have the absolute rigid types of materials and for small $\hat{g}$ we have the Bingham rheology closed to a Newtonian fluid. The last types of materials are closed to real rocks in the melted (strongly visco-plastic) parts of the Earth. Moreover, we see that the threshold of plasticity (yield point or yield limit) $\hat{g}$ is defined by the Mises' type relation

$$\tfrac{1}{2}\tau_{ij}^{D}\tau_{ij}^{D} \leqslant \hat{g}^2. \tag{13}$$

The momentum equation in Bussinesq's approximation, representing the classical approach for melting and solidification, is valid in the liquid zone, since the solid is assumed to be static. The additional term $M(f_S(c,T))\boldsymbol{v}$ in (1), based on empirical observations, describes the density variation in the mushy zone and means body forces. The density and the dynamic viscosity $\hat{\mu}$ are assumed to be piecewise constant. For the mushy zones the additional term $M(f_S)\boldsymbol{v}$ was found [1, 12] as

$$M(y) = C_0 y^2 (1 - y)^{-3}, \quad C_0 \text{ an empirical constant}, \tag{14}$$

and where the mushy zones are empirically modelled as a porous media. In Bingham's approximation both phases, liquid and solid, are represented by a strongly visco-plastic fluid, closed to viscous liquid and characterized by the lower value of the threshold of plasticity $\hat{g}$ in the first case, and by the viscous fluid with sufficiently high threshold of plasticity $\hat{g}$ in the second one, for which the volumic fractions are satisfied.

The diffusion equation for rock's liquid mixture (metallic's alloy) follows from the mass conservation law

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0. \tag{15}$$

For rock mixtures (metallic alloys) the definition of velocity must be defined in a new way.

If the diffusion is absent, then the composition of every given element of fluid remains unknown during its movement. This means that $\mathrm{d}c/\mathrm{d}t = 0$, i.e.,

$$\frac{\partial c}{\partial t} + \boldsymbol{u} \operatorname{grad} c = 0. \tag{16}$$

This and (15) yield

$$\frac{\partial(\rho c)}{\partial t} + \operatorname{div}(\boldsymbol{u}\rho c) = 0. \tag{17}$$

In the integral form (17) has the form

$$\frac{\partial}{\partial t}\int \rho c \,\mathrm{d}\boldsymbol{x} = -\oint \rho c\, \boldsymbol{u} \,\mathrm{d}s. \tag{18}$$

In the case if the diffusion is assumed, then

$$\frac{\partial}{\partial t}\int \rho c\,\mathrm{d}\boldsymbol{x} = -\oint \rho c\,\boldsymbol{u}\,\mathrm{d}s - \oint \boldsymbol{i}\,\mathrm{d}s, \tag{19}$$

where $\boldsymbol{i}$ denotes the density of diffusion flow. In the differential form (19) is as follows:

$$\frac{\partial(\rho c)}{\partial t} = -\mathrm{div}(\rho c\boldsymbol{u}) - \mathrm{div}\,\boldsymbol{i}. \tag{20}$$

From this and (15) we find

$$\rho\left(\frac{\partial c}{\partial t} + \boldsymbol{u}\,\mathrm{grad}\,c\right) = -\mathrm{div}\,\boldsymbol{i} \quad \text{in } \Omega\times I, \tag{21}$$

representing the equation of diffusion and where (see [20])

$$\boldsymbol{i} = -\rho D(\mathrm{grad}\,c + k_{\mathrm{T}}T^{-1}\,\mathrm{grad}\,T + k_p p^{-1}\mathrm{grad}\,p), \tag{22}$$

where D is the coefficient of diffusion, $k_{\mathrm{T}}D$ represents the coefficient of thermodiffusion and k_pD represents the coefficient of barodiffusion, T is the temperature, p is the pressure.

Finally, the energy conservation law gives the generalized heat equation in the form

$$\begin{aligned}&\rho c_e\left(\frac{\partial T(\boldsymbol{x},t)}{\partial t} + \boldsymbol{u}\,\mathrm{grad}\,T(\boldsymbol{x},t)\right) + \rho\beta_{ij}T_0 e_{ij}(\dot{\boldsymbol{u}})\\ &\quad= \mathrm{div}(\kappa(T)\,\mathrm{grad}\,T) + Q_0(\boldsymbol{x},t,T) \quad \text{in } \Omega_t.\end{aligned} \tag{23}$$

The coupled system of equations (1), together with (7), and (21)–(23) gives the governing equations for melted and solidified materials. The boundary and initial conditions must then describe the real situation of the investigated problem.

Remark. 'Liquid' phase shall mean the strongly visco-plastic material with a low threshold of plasticity $\hat{g}(\hat{g}\to 0)$ and a 'solid' phase shall mean the visco-plastic material with a threshold of plasticity $\hat{g}$ to be sufficiently large. Moreover, we can put $d=\rho D$, where e.g., $d\sim 10^{-9}$ for Al–Si (Al_2SiO_4) alloy. Furthermore, the liquid concentration c_{L} is the concentration c in the liquid zone ($c=f_{\mathrm{L}}c_{\mathrm{L}}+f_{\mathrm{S}}c_{\mathrm{S}}$, where f_{L}, f_{S} are the relative proportion of both phases-solid (visco-plastic) (S) and liquid (L), and $f_{\mathrm{L}}+f_{\mathrm{S}}=1$), c_{S} is the concentration c in solid visco-plastic phase. The diffusion factor d does not account for the different diffusivity of the solute in the liquid and solid (visco-plastic) phases. But both are very small, and the convective effect easily override the diffusive one (e.g., $d\sim 10^{-9}$ for Al–Si alloy). Moreover, compared to the heat diffusivity, the effect of the diffusion is much slower for the concentration. The body force term in (1) can be found as

$$\boldsymbol{F}(\boldsymbol{x},t,c,T) = \rho g x_3 + \mathrm{div}(\beta(T-T_0)) + \rho\boldsymbol{g}(c_0c_{\mathrm{L}}(\boldsymbol{x},t,c,T) + c_1T + c_2), \tag{24}$$

where the first term corresponds to body forces due to the effect of gravity, the second one represents thermal stresses and the third one represents the effect due to the concentration of 'liquid' phase (or gas) in the rocks. The effects of friction and Joule's heat can also be assumed.

We shall assume that the domain investigated $\overline{\Omega} = \bigcup_{\iota=1}^{m}\bigcup_{s=1}^{r}\overline{\Omega}^{\iota,s}$, where Ω^{ι} denotes subdomains of Ω characterized by the material properties and indices 's' characterize the domains with the phase changes of rocks (melting, recrystallization, solidification, respectively).

3. Mathematical formulation of the problem

Let $\Omega = \bigcup_{\iota=1}^{m} \Omega^{\iota} \subset \mathbb{R}^N$, $N = 2(3)$ be a bounded domain occupied by the interior of the Earth or its parts, or by a metal body in technological practice, respectively, with a smooth boundary $\partial\Omega = \Gamma_u \cup \Gamma_\tau \cup \mathscr{R}$, $\mathscr{R}$ the set of zero measure. Let $I = (t_0, t_1)$ and let $\Omega_T = \Omega \times \bar{I}$, $\partial\Omega_T = \partial\Omega \times \bar{I}$, $\Omega_t = \Omega \times (t_0, t)$, $\partial\Omega_t = \partial\Omega \times (t_0, t)$, $t \in I$. We suppose that the components of $\partial\Omega$ are smooth enough to admit a normal $\boldsymbol{n}$ almost everywhere in the sense of the surface measure on $\partial\Omega$. Moreover, we shall assume incompressible materials. Furthermore, we shall assume that $\Gamma^{\iota,s}$, $\iota = 1,\dots,m$, $s = 1,\dots,r$, denote $n-1$ dimensional open sets in the relative topology of Ω, i.e. surfaces which divide Ω into r open sets $\Omega^{\iota,s}$. We denote the respective portions of the boundaries of $\Omega^{\iota,s}$ and $\Omega^{\iota,s+1}$ by $\partial\Omega^{\iota,s}$ and $\partial\Omega^{\iota,s+1}$. We shall assume that the components $\Gamma^{\iota,s} \equiv R^{\iota,s}(t)$ of $\Gamma = \bigcup_{\iota,s} \Gamma^{\iota,s}$ are smooth enough to also admit normals $\boldsymbol{v}^{\iota,s}$ almost everywhere in the sense of the surface measures on $\Gamma^{\iota,s} \equiv R^{\iota,s}(t)$. We shall assume that $\Omega^{\iota,s}$, $\Omega^{\iota,s+1}$ constitutes other state of materials, i.e., 'solid', 'liquid' or other types of recrystallized rocks. Thus $\Gamma^{\iota,s} \equiv R^{\iota,s}(t)$ denote the interfaces between two different phase states of rocks and we speak about the phase change boundaries or phase transitient boundaries (zones). Here $\Omega^{\iota,s}$, $\Omega^{\iota,s+1}$, $\Gamma^{\iota,s}$ are unknowns that must be determined as part of the solution. We denote by $T^{\iota,s}(\boldsymbol{x},t)$ the temperature of rocks in $\Omega^{\iota,s}$ and by $\boldsymbol{u}^{\iota,s}(\boldsymbol{x},t)$ and $p^{\iota,s}(\boldsymbol{x},t)$ the velocity and pressure in $\Omega^{\iota,s}$ and by $c^{\iota,s}(\boldsymbol{x},t)$ and $\varphi^{\iota,s}(\boldsymbol{x},t)$ the concentrations and sources of concentrations in $\Omega^{\iota,s}$.

Then we shall investigate the nonstationary incompressible coupled two-phase Stefan–Bingham problem:

Problem ($\mathscr{P}_0$). Consider the problem of finding functions $T(\boldsymbol{x},t)$, $\boldsymbol{u}(\boldsymbol{x},t)$, $p(\boldsymbol{x},t)$ and $c(\boldsymbol{x},t)$ defined on the clossure $\overline{\Omega}^{\iota,s}$ of open sets $\Omega^{\iota,s}$ such that $\Omega = \bigcup_{\iota,s} \overline{\Omega}^{\iota,s}$ and satisfying

$$\rho^{\iota} c_e^{\iota}\left(\frac{\partial T^{\iota}}{\partial t} + u_k^{\iota}\frac{\partial T^{\iota}}{\partial x_k}\right) + \rho^{\iota}\beta_{ij}^{\iota} T_0^{\iota} e_{ij}(\dot{\boldsymbol{u}}^{\iota}) = Q_0^{\iota}(\boldsymbol{x},t) + \frac{\partial}{\partial x_i}\left(\kappa_{ij}^{\iota}\frac{\partial T^{\iota}}{\partial x_j}\right) \tag{25}$$

in $\Omega^{\iota} \times I$, $i,j = 1,2(3)$,

$$\rho^{\iota}\left(\frac{\partial u_i^{\iota}}{\partial t} + u_k^{\iota}\frac{\partial u_i^{\iota}}{\partial x_k}\right) = \frac{\partial \tau_{ij}^{\iota}}{\partial x_j} + F_i^{\iota} \quad \text{in } \Omega^{\iota} \times I, \quad i,j = 1,2(3), \tag{26}$$

$$\operatorname{div} \boldsymbol{u}^{\iota} = 0 \quad \text{in } \Omega^{\iota} \times I, \tag{27}$$

$$\frac{\partial c^{\iota}}{\partial t} + \boldsymbol{u}^{\iota} \operatorname{grad} c^{\iota} = \lambda^{\iota} \Delta c^{\iota} + \varphi^{\iota} \quad \text{in } \Omega^{\iota} \times I, \tag{28}$$

and boundary and initial conditions

$$T(\boldsymbol{x},t) = T_1(\boldsymbol{x},t)(=0), \quad \tau_{ij}(\boldsymbol{x},t)n_j = P_{0i}(\boldsymbol{x},t), \quad \frac{\partial c(\boldsymbol{x},t)}{\partial n} = 0 \tag{29}$$

on $\Gamma_\tau \times I$, $i,j = 1,2(3)$,

$$\kappa_{ij}\frac{\partial T(\boldsymbol{x},t)}{\partial x_j} n_i = q(\boldsymbol{x},t), \quad \boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_1(\boldsymbol{x},t), \quad \frac{\partial c(\boldsymbol{x},t)}{\partial \boldsymbol{n}} = 0 \tag{30}$$

on $\Gamma_u \times I$, $i,j = 1,2(3)$,

$$T_{\rm S}^{\iota,s} = T_{\rm L}^{\iota,s} = T_{\rm R}^{\iota,s}, \quad \left(\kappa_{ij}^{\iota}\frac{\partial T^{\iota}}{\partial x_j}v_i^{\iota}\right)_{\rm S}^{s} - \left(\kappa_{ij}^{\iota}\frac{\partial T^{\iota}}{\partial x_j}v_i^{\iota}\right)_{\rm L}^{s} = -\rho^{\iota,s}L^{\iota,s}v_v^{\iota,s} \tag{31}$$

on $R^{\iota,s}(t)$,

$$T^{\iota}(\boldsymbol{x},t_0) = T_0^{\iota}(\boldsymbol{x}), \quad \boldsymbol{u}^{\iota}(\boldsymbol{x},t_0) = \boldsymbol{u}_0^{\iota}(\boldsymbol{x}), \quad c^{\iota}(\boldsymbol{x},t_0) = c_0^{\iota}(\boldsymbol{x}) \quad \text{in } \Omega^{\iota}, \tag{32}$$

where τ_{ij} is defined by the constituent law (7) and where $\rho^{\iota,s}$ is a density associated with $\Omega^{\iota,s}$, $T_{\rm S}^{\iota,s}$, $T_{\rm L}^{\iota,s}$, $T_{\rm R}^{\iota,s}$ are temperatures in $\Omega^{\iota,s}$, where rocks (metals) are in a solid-visco-plastic state ($T_{\rm S}^{\iota,s}$) or in a liquid (strongly visco-plastic with low threshold of plasticity $\hat{g}$) state ($T_{\rm L}^{\iota,s}$) or temperatures of phase changes, resp., $\hat{v}^{\iota,s}$ is a normal to the phase change boundary $R^{\iota,s}(t) \equiv \Gamma^{\iota,s}$ pointing towards $\Omega_{\rm S}^{\iota,s}$, $v_v^{\iota,s} = \partial\Phi^{\iota,s}/\partial t \equiv \partial_t\Phi^{\iota,s}$ are the speeds of $R^{\iota,s}(t)$ along $\boldsymbol{v}^{\iota,s}$, $L^{\iota,s}$ are the latent heats of the phase changes (melting, recrystallization, solidification) (The latent heat represents the energy necessary for a complete phase change of a unit volume at melting temperature and null concentration. It is positive in the case of melting and recrystallization and negative in the case of solidification) and $R^{\iota,s}(t)$ represent hypersurfaces lying in $\Omega_t^{\iota,s}$, $\Omega_{{\rm S}t}^{\iota,s}$ are domains lying in $\Omega_t^{\iota,s}$ and bounded by $R^{\iota,s}(t)$ and $\partial\Omega_t^{\iota,s}$. Furthermore, let $\Phi^{\iota,s}(\boldsymbol{x},t)$ be a C^1 function in $\overline{\Omega}_T$ such that

$$R^{\iota,s}(t) = \{(\boldsymbol{x},t) \in \overline{\Omega}_T, \Phi^{\iota,s}(\boldsymbol{x},t) = 0\}, \quad \nabla_x\Phi^{\iota,s}(\boldsymbol{x},t) \neq 0 \quad \text{on } R^{\iota,s}(t),$$

$$\Phi^{\iota,s}(\boldsymbol{x},t) < 0 \quad \text{in } \Omega_{\rm S}^{\iota,s} \times I, \quad \Phi^{\iota,s}(\boldsymbol{x},t) > 0 \quad \text{in } \Omega_{\rm L}^{\iota,s} \times I$$

and

$$R(t) = \bigcup_{\iota,s} R^{\iota,s}(t), \quad t_0 \leqslant t < t_1$$

is the phase change or free boundary. The functions $T_0^{\iota}(\boldsymbol{x})$, $T_1^{\iota}(\boldsymbol{x},t)$, $\boldsymbol{u}_0(\boldsymbol{x})$, $\boldsymbol{u}_1(\boldsymbol{x},t)$, $c_0(\boldsymbol{x})$ are the initial and boundary data on $\Gamma_\tau \times I$ or $\Gamma_u \times I$, respectively, $c_{\rm e}^{\iota}(\boldsymbol{x})$ are specific heat in Ω^{ι}, $\kappa_{ij}^{\iota}(\boldsymbol{x})$ are the thermal conductivity in Ω^{ι} and $Q^{\iota}(\boldsymbol{x},t)$, $\varphi^{\iota}(\boldsymbol{x},t)$ are thermal sources and sources of concentration, respectively, in $\Omega^{\iota} \times I$, $\iota = 1,\ldots,m$. All the functions used in this paper are real values. Further, we shall investigate the 2D model problem ($N = 2$) only.

To analyze the Stefan problem we shall introduce a new variable Θ by the Kirchhoff transformation

$$\Theta = G(\boldsymbol{x},t,T) = \int_0^{T(x,t)} \kappa(\boldsymbol{x},t,\xi)\,{\rm d}\xi, \quad (\boldsymbol{x},t) \in \overline{\Omega}_t.$$

Since $\kappa_0 \leqslant \kappa(\boldsymbol{x},t,\xi) \leqslant \kappa_1$ for $(\boldsymbol{x},t,\xi) \in \overline{\Omega} \times I \times R$, $\kappa_0, \kappa_1 = \text{const.} > 0$, then the mapping $T \to \Theta(\boldsymbol{x},t,T)$ is one-to-one and $T = G^{-1}(\boldsymbol{x},t,\Theta)$, $G^{-1}(\boldsymbol{x},t,\Theta)$ is its inverse. We introduce the enthalpy by

$$H(\boldsymbol{x},t,T) = \int_0^{T(x,t)} \underline{c}((G^{-1}(\zeta))\kappa^{-1}(G^{-1}(\zeta)))\,{\rm d}\zeta + \begin{cases} 0 & \text{if } T < \Theta_R(\boldsymbol{x},t), \\ \langle 0,\alpha\rangle & \text{if } T = \Theta_R(\boldsymbol{x},t), \\ \alpha & \text{if } T > \Theta_R(\boldsymbol{x},t), \end{cases}$$

where we denoted by $\underline{c}(\boldsymbol{x},t) = \rho(\boldsymbol{x},t)c_{\rm e}(\boldsymbol{x},t)$, and $\Theta_R(\boldsymbol{x},t) = G(\boldsymbol{x},t,T_R)$, T_R is the temperature of a phase change, $\alpha = \rho L$.

Then we shall investigate the nonstationary incompressible coupled two-phase Stefan in the Bingham rheology and in the enthalpy formulation:

Problem ($\mathscr{P}$). Consider the problem of finding functions $H^{\iota,s}(\boldsymbol{x},t)$, $\Theta^{\iota,s}(\boldsymbol{x},t)$, $c^{\iota,s}(\boldsymbol{x},t)$, $\boldsymbol{u}^{\iota,s}(\boldsymbol{x},t)$ (and $p^{\iota,s}(\boldsymbol{x},t)$) defined on the clossure $\overline{\Omega}^{\iota,s}$ of open sets $\Omega^{\iota,s}$ such that $\Omega=\bigcup_{\iota,s}\overline{\Omega}^{\iota,s}$ and satisfying

$$\left(\frac{\partial H^\iota}{\partial t}+u_k^\iota\frac{\partial\Theta^\iota}{\partial x_k}\right)+\rho^\iota\beta_{ij}^\iota\Theta_0^\iota e_{ij}(\boldsymbol{u}^\iota)=Q^\iota(\boldsymbol{x},t)+\Delta\Theta^\iota(\boldsymbol{x},t)\ \text{in}\quad\Omega^\iota\times I, \tag{33}$$

$$\rho^\iota\left(\frac{\partial u_i^\iota(\boldsymbol{x},t)}{\partial t}+u_k^\iota\frac{\partial u_i^\iota(\boldsymbol{x},t)}{\partial x_k}\right)=\frac{\partial\tau_{ij}^\iota(\boldsymbol{x},t)}{\partial x_j}+F_i^\iota(\boldsymbol{x},t)\ \text{in}\ \Omega^\iota\times I,\quad i,j=1,2, \tag{34}$$

$$\operatorname{div}\boldsymbol{u}^\iota=0\quad\text{in}\ \Omega^\iota\times I, \tag{35}$$

$$\frac{\partial c^\iota(\boldsymbol{x},t)}{\partial t}+\boldsymbol{u}^\iota(\boldsymbol{x},t)\operatorname{grad}c_{\mathrm{L}}^\iota(c^\iota(\boldsymbol{x},t),\Theta^\iota(\boldsymbol{x},t))=\lambda^\iota(\boldsymbol{x})\Delta c^\iota(\boldsymbol{x},t)+\varphi^\iota(\boldsymbol{x},t)\quad\text{in}\ \Omega^\iota\times I, \tag{36}$$

$$\Theta^\iota(\boldsymbol{x},t)=T^\iota(c^\iota(\boldsymbol{x},t),H^\iota(\boldsymbol{x},t))\quad\text{in}\ \Omega^\iota\times I, \tag{37}$$

and boundary and initial conditions

$$H(\boldsymbol{x},t)=0,\quad\tau_{ij}(\boldsymbol{x},t)n_j=P_{0i}(\boldsymbol{x},t),\quad\frac{\partial c(\boldsymbol{x},t)}{\partial\boldsymbol{n}}=0 \tag{38}$$

on $\Gamma_\tau\times I$,

$$\frac{\partial\Theta(\boldsymbol{x},t)}{\partial n}=q(\boldsymbol{x},t),\quad\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{u}_1(\boldsymbol{x},t),\quad\frac{\partial c(\boldsymbol{x},t)}{\partial\boldsymbol{n}}=0 \tag{39}$$

on $\Gamma_u\times I$,

$$H^\iota(\boldsymbol{x},t)=H_0^\iota(\boldsymbol{x}),\quad\boldsymbol{u}^\iota(\boldsymbol{x},t_0)=\boldsymbol{u}_0^\iota(\boldsymbol{x}),\quad c^\iota(\boldsymbol{x},t_0)=c_0^\iota(\boldsymbol{x}) \tag{40}$$

in Ω^ι, where

$$\tau_{ij}=-p\delta_{ij}+\hat{g}D_{ij}D_{\mathrm{II}}^{-1/2}+2\hat{\mu}D_{ij}+\beta_{ij}(T-T_0) \tag{41}$$

and where $D_{ij}=\frac{1}{2}(\partial_j u_i+\partial_i u_j)$, $D_{\mathrm{II}}=\frac{1}{2}D_{ij}D_{ij}$, and $H_0^\iota(\boldsymbol{x})$, $c_0^\iota(\boldsymbol{x})$, $\lambda^\iota(\boldsymbol{x})$, $\boldsymbol{u}_0^\iota(\boldsymbol{x})$ are the given positive functions. For simplicity we shall assume that $\boldsymbol{u}_1(\boldsymbol{x},t)=0$.

In the next we shall also use the following notation: $\partial_t r=\partial r/\partial t$, where r is a scalar or vector function, respectively and $\partial_s r(y_1,y_2)=\partial r(y_1,y_2)/\partial y_s$, $s=1,2$, and similarly, $\partial_j v_i=v_{i,j}=\partial v_i/\partial x_j$. Further, for simplicity indices ι and s will be omitted.

Furthermore, we shall assume that

(A1) $\Omega=\bigcup_{\iota=1}^m\Omega^\iota\subset\mathbb{R}^2$ is an open bounded connected domain with Lipschitzian boundary $\partial\Omega=\Gamma_u\cup\Gamma_\tau\cup\mathscr{R}$.

On the domain Ω for functions T, c_{L}, multipliers m and the initial and boundary data we shall assume:

(A2) For the function $T^\iota : \mathbb{R}_+^2 \to \mathbb{R}$, where $\mathbb{R}_+$ denotes the set of non-negative reals, we assume that
(i) T^ι is uniformly Lipschitzian,[3]
(ii) $T^\iota(x,0)=0\ \forall x\in\mathbb{R}_+$,
(iii) open sets (possibly empty) $O_i\subset\mathbb{R}_+^2$, $i\in\mathcal{N}$ exist, such that T^ι is constant on each O_i and such that a function $\alpha^\iota\in C^0(\langle 0,\infty))\cap C^2((0,\infty))$ exists, verifying:
$(\alpha'')^\iota$ is decreasing, $\lim_{x\to\infty}(\alpha''(x))^\iota>0$, and $\partial_2 T^\iota(x,y)\geqslant((\alpha''(x))^\iota)^{-1}$ for a.e. $(x,y)\in\mathbb{R}_+^\iota\times\mathbb{R}_+\setminus\bigcup_{i\in\mathcal{N}}O_i$.
(A3) Since the term $\hat{g}^\iota D_{ij}^\iota(\boldsymbol{u}^\iota)D_{\mathrm{II}}^{-1/2}(\boldsymbol{u}^\iota)$ creates in the variational formulation the nondifferential functional, then we introduce the multipliers $m^\iota=(m_{ij}^\iota)$, $i,j=1,2$, by
(i) $m_{ij}^\iota\in L^\infty(\Omega^\iota\times I)$, $m_{ij}^\iota=m_{ji}^\iota$, $\forall i,j=1,2$, $m_{ii}^\iota=0$,
(ii) $m_{ij}^\iota m_{ij}^\iota\leqslant 1$ a.e. in $\Omega^\iota\times I$,
(iii) $m_{ij}^\iota D_{ij}(\boldsymbol{u}^\iota)=(D_{ij}(\boldsymbol{u}^\iota)D_{ij}(\boldsymbol{u}^\iota))^{1/2}=2^{1/2}D_{\mathrm{II}}^{1/2}(\boldsymbol{u}^\iota)$ a.e. in $\Omega^\iota\times I$.
(iv) Then the constituent law (7) can be rewritten as

$$\tau_{ij}^\iota = -p^\iota\delta_{ij}+2\hat{\mu}^\iota D_{ij}(\boldsymbol{u}^\iota)+2^{1/2}\hat{g}^\iota m_{ij}^\iota-\beta_{ij}^\iota(T^\iota-T_0^\iota).$$

(A4) The functions $c_{\mathrm{L}}^\iota:\mathbb{R}_+^2\to\mathbb{R}$, representing a concentration in a liquid (strongly visco-plastic with low $\hat{g}$) part of the medium and stem from the constitutive law (phase diagram) satisfy
(i) $c_{\mathrm{L}}^\iota\in C^0(\mathbb{R}_+^2)\cap L^\infty(\mathbb{R}_+^2)$,
(ii) $c_{\mathrm{L}}^\iota(0,y)=0\ \forall y\in\mathbb{R}_+$,
(iii) $\exists c_1=\text{const.}>0$ such that $|c_{\mathrm{L}}^\iota(x,y)|\leqslant c_1[(\alpha''(x))^\iota]^{-1/2}\ \forall(x,y)\in R^\iota\times\mathbb{R}_+$, where α^ι satisfies (A2) (iii).
(A5) The initial data satisfy
(i) $H_0(\boldsymbol{x})\in L^2(\Omega)$, $H_0(\boldsymbol{x})\geqslant 0$ a.e in Ω,
(ii) $\boldsymbol{u}_0(\boldsymbol{x})\in H(\Omega)=\{\boldsymbol{v}|\boldsymbol{v}\in[L^2(\Omega)]^2,\ \operatorname{div}\boldsymbol{v}=0,\ \boldsymbol{v}=0 \text{ on } \Gamma_u\}$, $\boldsymbol{u}_0(\boldsymbol{x})\geqslant 0$ a.e. in Ω,
(iii) $c_0(\boldsymbol{x})\in L^2(\Omega)$, $c_0(\boldsymbol{x})\geqslant 0$ a.e. in Ω.
(A6) The physical data is sufficiently smooth, i.e.

$$\begin{array}{ll}
Q^\iota(\boldsymbol{x},t)\in L^2(I;L^2(\Omega^\iota)), & \boldsymbol{F}^\iota(\boldsymbol{x},t)\in L^2(I;[L^2(\Omega^\iota)]^2),\\
\varphi^\iota(\boldsymbol{x},t)\in L^2(I;L^2(\Omega^\iota)), & q(\boldsymbol{x},t)\in L^2(I;L^2(\Gamma_u)),\\
\boldsymbol{P}(\boldsymbol{x},t)\in L^2(I;[L^2(\Gamma_\tau)]^2), & \kappa^\iota(\boldsymbol{x},t)\in L^2(I;C^1(\overline{\Omega}^\iota)),\\
\hat{\mu}^\iota(\boldsymbol{x}),\hat{g}^\iota(\boldsymbol{x})=\text{const.}>0, & \partial\beta_{ij}^\iota(\boldsymbol{x})/\partial x_j\in L^\infty(\Omega^\iota)\quad\forall i,j,\\
\lambda^\iota(\boldsymbol{x})\in C^1(\overline{\Omega}^\iota). &
\end{array}$$

Remark. The conditions (iii) in assumptions (A2) and (A4) are satisfied if positive constants c_1, c_2 and a real $\delta\in\langle 0,2)$ exist such that (the symbol 'ι' is omitted)
(i) $\partial_2 T(x,y)\geqslant c_1\min(x^\delta,1)$ for almost every $(x,y)\in\mathbb{R}_+^2\setminus\bigcup_{i\in\mathcal{N}}O_i$ and
(ii) $|c_{\mathrm{L}}(x,y)|\leqslant c_2\min(x^{\frac{1}{2}\delta},1)\ \forall(x,y)\in\mathbb{R}_+^2$. In that case, the function α is such that

[3] Let M be a measurable subset of $\mathbb{R}^N$ and let $T:M\to\mathbb{R}^N$ be a mapping satisfying the Lipschitz condition on M with a constant $\alpha>0$, i.e. $|T(x)-T(y)|\leqslant\alpha|x-y|\ \forall x,y\in M$.

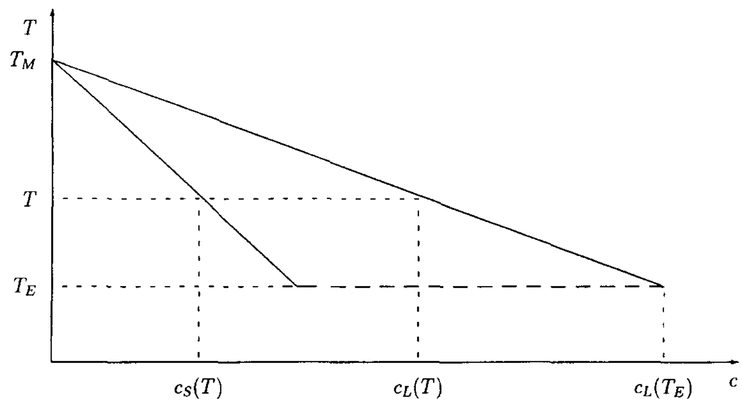

Fig. 1. Phase diagram (c, T) (simple version).

(iii)

$$\alpha''(x) = \begin{cases} c^{-1}x^{-\delta}, & x<1, \\ c_1^{-1}, & x \geqslant 1. \end{cases}$$

Because of assumption (A2) (iii), the system of equations for determining $T(\boldsymbol{x},t)$ and $c(\boldsymbol{x},t)$ is generally degenerated and Eq. (25) is similar to classical Stefan's equation.

Remark. Function c in the mushy zone can be written as (see the phase diagram) $c = c_{\mathrm{L}} f_{\mathrm{L}} + c_{\mathrm{S}} f_{\mathrm{S}} = c_{\mathrm{L}}(1 - f_{\mathrm{S}}) + c_{\mathrm{S}} f_{\mathrm{S}}$, where f_{L}, f_{S} represent the relative portion of visco-plastic (solid) and strongly visco-plastic with low $\hat{g}$ (liquid) phases, respectively and will be determined as a result of computations, where the portion of 'solid' and 'liquid' parts are controlled means of the threshold of plasticity $\hat{g}$. In the phase diagram T_{M} is the temperature of melting, T_{E} is the eutectic temperature, under which no 'liquid' remains; $c_{\mathrm{S}}(T)$ is the concentration of the solute for which the 'solid' rock begins to melt, $c_{\mathrm{L}}(T)$ is the concentration of the solute for which the 'liquid' rock begins to solidify. The mushy zone corresponds to points (c, T), where $c \in \langle c_{\mathrm{S}}(T), c_{\mathrm{L}}(T)\rangle$ and $T \in \langle T_{\mathrm{E}}, T_{\mathrm{M}}\rangle$.

4. Variational formulation

Let us denote by $C^k(\Omega)$, $0 \leqslant k < \infty$, the space of all functions f defined on Ω which have continuous derivatives up to the order k on Ω; for $k=0$ we put $C^0(\Omega) = C(\Omega)$. We denote by $H^1(\Omega)$ the Sobolev space of scalar (or vector, respectively) functions such that $f \in L^2(\Omega)$, $\partial_i f \in L^2(\Omega)$, $i = 1, \ldots, N$, where $L^2(\Omega)$ is the space of square summable scalar (or vector, respectively) functions on Ω and the derivatives $\partial_i f$ are taken in the sense of distributions on Ω. By $L^\infty(\Omega)$ we denote the space of all measurable functions f defined a.e. on Ω so that a constant $c > 0$ exists such that

$$|f(\boldsymbol{x})| \leqslant c \quad \forall \boldsymbol{x} \in \Omega \backslash E, \quad E \subset \Omega, \quad \mathrm{meas}(E) = 0.$$

By $L^p(I;X)$, where X is a functional space, $1 \leqslant p < \infty$, we denote the space of functions $f: I \to X$ such that $\|f(.)\|_X \in L^p(I)$ and by $H^1(I;X)$ the space of functions $f: I \to X$ for which $\|f(.)\|_X \in H^1(I)$.

Let us define the spaces

$$^1V=\{z\,|\,z\in H^1(\Omega), z=0 \text{ on } \Gamma_\tau\},\qquad {}^2V=\{w\,|\,w\in H^1(\Omega)\},$$

$$V=\{\boldsymbol{v}\,|\,\boldsymbol{v}\in[H^1(\Omega)]^2,\ \operatorname{div}\boldsymbol{v}=0, \boldsymbol{v}=0 \text{ on } \Gamma_u\},$$

$$H(\Omega)=\{\boldsymbol{v}\,|\,\boldsymbol{v}\in[L^2(\Omega)]^2,\ \operatorname{div}\boldsymbol{v}=0,\ \boldsymbol{v}=0 \text{ on } \Gamma_u\},$$

$^1V'$, $^2V'$, V' be dual spaces of ^1V, ^2V and V, respectively,

$$^1V\subset L^2(\Omega)\subset {}^1V',\qquad {}^2V\subset L^2(\Omega)\subset {}^2V',\qquad V\subset H\subset V',$$

$$^1V'\equiv H^{-1}(\Omega),\qquad {}^2V'\equiv (H^1(\Omega))',$$

$$^1W=\{z\,|\,z\in L^2(I;{}^1V), z(\boldsymbol{x},t_0)=0\},\qquad {}^2W=\{w\,|\,w\in L^2(I;{}^2V), w(\boldsymbol{x},t_0)=0\},$$

$$W=\{\boldsymbol{v}|\boldsymbol{v}\in L^2(I;V), \boldsymbol{v}'\in L^2(I;H), \boldsymbol{v}(\boldsymbol{x},t_0)=0\}.$$

Then we have the following variational (weak) formulation of the coupled problem defined in enthalpy:

Problem ($\mathscr{P}_{\text{centh}}$). Find a tetrad $\{H(\boldsymbol{x},t),\Theta(\boldsymbol{x},t),\boldsymbol{u}(\boldsymbol{x},t),c(\boldsymbol{x},t)\}$, i.e., enthalpy H, generalized temperature Θ, velocity $\boldsymbol{u}$ and concentration c, satisfying

$$\int_I\{(\partial_t H(t), z-\Theta(t))+(\boldsymbol{u}(t)\operatorname{grad}\Theta, z-\Theta(t))+a_\Theta(\Theta(t), z-\Theta(t))$$
$$+b_p(\boldsymbol{u}(t), z-\Theta(t))-(Q(t), z-\Theta(t))\}\,\mathrm{d}t\geqslant 0\quad \forall z\in {}^1W, \tag{42}$$

$$\int_I\{(\partial_t\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))+\hat{\mu}a(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))+b(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))$$
$$+\hat{g}j(\boldsymbol{v})-\hat{g}j(\boldsymbol{u}(t))+b_s(\Theta(t)-\Theta_0,\boldsymbol{v}-\boldsymbol{u}(t))$$
$$-(\boldsymbol{F}(t),\boldsymbol{v}-\boldsymbol{u}(t))\}\,\mathrm{d}t\geqslant 0\quad \forall\boldsymbol{v}\in W, \tag{43}$$

$$\int_I\{(\partial_t c(t), d-c(t))+(\boldsymbol{u}(t)\operatorname{grad} c_{\mathrm{L}}(c,\Theta), d-c(t))$$
$$+a_c(c(t), d-c(t))-(\varphi(t), d-c(t))\}\,\mathrm{d}t\geqslant 0,\quad \forall d\in {}^2W, \tag{44}$$

$$\Theta(t)=T(c(t),H(t)),\quad H\geqslant 0,\ c\geqslant 0 \text{ a.e. in } \Omega\times I, \tag{45}$$

$$H(\boldsymbol{x},t_0)=H_0(\boldsymbol{x}),\quad \boldsymbol{u}(\boldsymbol{x},t_0)=\boldsymbol{u}_0(\boldsymbol{x}),\quad c(\boldsymbol{x},t_0)=c_0(\boldsymbol{x})\quad \text{a.e. in } \Omega, \tag{46}$$

where $H^{-1}(\Omega)$ stands for $(H_0^1(\Omega))'$ and where $\boldsymbol{u},\boldsymbol{v}\in[H^1(\Omega)]^2$, Θ, z, c and $d\in H^1(\Omega)$, $\varphi, Q\in L^2(\Omega)$, $\boldsymbol{F}\in[L^2(\Omega)]^2$ and

$$a_\Theta(\Theta,z)=\int_\Omega \operatorname{grad}\Theta\operatorname{grad} z\,\mathrm{d}\boldsymbol{x},\qquad a(\boldsymbol{u},v)=2\int_\Omega D_{ij}(\boldsymbol{u})D_{ij}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x},$$

$$a_c(c,d)=\int_\Omega \lambda\operatorname{grad} c\operatorname{grad} d\,\mathrm{d}\boldsymbol{x},\qquad b_\Theta(\boldsymbol{u},\Theta,z)=(\boldsymbol{u}\operatorname{grad}\Theta,z),$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_\Omega \rho u_i v_{j,i} w_j \,\mathrm{d}\boldsymbol{x}=(\rho\boldsymbol{u}\ \mathrm{grad}\ \boldsymbol{v},\boldsymbol{w}),$$

$$b_s(\Theta,\boldsymbol{v})=\int_\Omega \frac{\partial}{\partial x_j}(\beta_{ij}\Theta)v_i\,\mathrm{d}\boldsymbol{x},\qquad b_p(\boldsymbol{v},g)=\int_\Omega \rho\Theta_0\beta_{ij}v_{i,j}g\,\mathrm{d}\boldsymbol{x},$$

$$b_c(\boldsymbol{u},c,d)=(\boldsymbol{u}\ \mathrm{grad}\ c,d),\qquad (\boldsymbol{u},\boldsymbol{v})=\int_\Omega u_iv_i\,\mathrm{d}\boldsymbol{x},\qquad (\Theta,z)=\int_\Omega \Theta z\,\mathrm{d}\boldsymbol{x},$$

$$(Q,z)=\int_\Omega Qz\,\mathrm{d}\boldsymbol{x}-\int_{\Gamma_u} qz\,\mathrm{d}s,\quad (\boldsymbol{F},\boldsymbol{v})=\int_\Omega F_iv_i\,\mathrm{d}\boldsymbol{x}-\int_{\Gamma_\tau} P_{0i}v_i\,\mathrm{d}\boldsymbol{x},$$

$$j(\boldsymbol{v})=2\int_\Omega (D_{\mathrm{II}}(\boldsymbol{v}))^{1/2}\,\mathrm{d}\boldsymbol{x}.$$

Definition 1. We said that the tetrad $\{H,\Theta,\boldsymbol{u},c\}$ is a weak solution of the problem $(\mathscr{P}_{\mathrm{centh}})$ if $H\in L^\infty(I;L^2(\Omega))\cap H^1(I;H^{-1}(\Omega))$, $\Theta\in L^2(I;{}^1V)$, $\boldsymbol{u}\in L^\infty(I;L^2(\Omega))\cap L^2(I;V)$, $c\in L^2(I;{}^2V)\cap H^1(I;H^1(\Omega)')$, $H\geqslant 0$, $c\geqslant 0$ and $\Theta=T(c,H)$ a.e. in $\Omega\times I$ and (42)–(44), (46) hold.

4.1. Preliminary results and main theorem

In the sequel the following lemmas will be used:

Lemma 1. *It holds*

$$v_{i,j}\in L^2(\Omega)\quad \forall \boldsymbol{v}\in V,\ i,j=1,2.$$

Proof. If $\boldsymbol{v}\in V$, then $v_{i,j}\in L^2(\Omega)$. The proof is a consequence of Lemma 3.1 of [7].

Lemma 2 (Gronwall). *Let $g(t)\in C(I)$, $g(t)\geqslant 0$, $\rho(t)\in C(I)$, $\rho(t)\geqslant 0$, $g(t)$ be the nondecreasing function with increasing t.*

Let $\rho(t)$ be a solution of the inequality

$$\rho(t)\leqslant c_0\int_{t_0}^t \rho(\tau)\,\mathrm{d}\tau+g(t),\quad t_0\leqslant t\leqslant t_1,\ c_0=\mathrm{const.}$$

Then $c_1=\mathrm{const.}$, $c_1=c_1(c_0,t_0,t_1)$ exists such that

$$\rho(t)\leqslant c_1 g(t)\quad \forall t,\ t_0\leqslant t\leqslant t_1.$$

For proof see [22].

Further, we have the following estimates: Firstly, from the above given assumptions, the symmetry conditions

$$a_\Theta(\Theta,z)=a_\Theta(z,\Theta),\qquad a(\boldsymbol{u},\boldsymbol{v})=a(\boldsymbol{v},\boldsymbol{u}),\qquad a_c(c,d)=a_c(d,c)$$

hold. Moreover, it yield that for $\Theta \in {}^1V$, $\boldsymbol{u} \in V$, $c \in {}^2V$ constant $c_\Theta > 0$, $c_u > 0$, $c_c > 0$ exist such that

$$a_\Theta(\Theta,\Theta) \geqslant c_\Theta \|\Theta\|_{1,1}^2 \quad \forall \Theta \in {}^1V, \quad a(\boldsymbol{u},\boldsymbol{u}) \geqslant c_u \|\boldsymbol{u}\|_{1,2}^2 \quad \forall \boldsymbol{u} \in V,$$

$$a_c(c,c) \geqslant c_c \|c\|_{1,1}^2 \quad \forall c \in {}^2V.$$

For $\boldsymbol{u}$, y, z it holds $b_\Theta(\boldsymbol{u},y,z) + b_\Theta(\boldsymbol{u},z,y) = 0$, $\boldsymbol{u} \in H$, y, $z \in H_0^1(\Omega)$ and positive constants c_5, c_6, c_7 independent of $\boldsymbol{u}$, y, z exist such that

$$\begin{aligned} |b_\Theta(\boldsymbol{u},y,z)| &\leqslant c_5 \|\boldsymbol{u}\|_{[L^p(\Omega)]^2} \|y\|_{L^p(\Omega)} \|\{D_i z\}\|_{0,1} \\ &\leqslant c_6 \|\boldsymbol{u}\|_{[L^p(\Omega)]^2} \|y\|_{L^p(\Omega)} \|z\|_{1,1} \\ &\leqslant c_7 \|\boldsymbol{u}\|_{1,2}^{\frac{1}{2}} \|\boldsymbol{u}\|_{0,2}^{\frac{1}{2}} \|y\|_{1,1}^{\frac{1}{2}} \|y\|_{0,1}^{\frac{1}{2}} \|z\|_{1,1} \end{aligned}$$

and similarly for $b_c(\boldsymbol{u},c,d)$.

For a vector field $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ on Ω we put

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega u_i v_{i,j} w_i \,\mathrm{d}\boldsymbol{x}$$

and $|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant c \|\boldsymbol{u}\|_{[L^p(\Omega)]^2} \|\boldsymbol{w}\|_{[L^p(\Omega)]^2} \sum_{i,j} \|D_i v_j\|_{0,2}$, $(2/p) + \frac{1}{2} = 1$.

Using Lemma 1 we find that

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant c \|\boldsymbol{u}\|_{[L^p(\Omega)]^2} \|\boldsymbol{w}\|_{[L^p(\Omega)]^2} \|\boldsymbol{v}\|_{1,2}.$$

Moreover, the inequality of convexity

$$\|\boldsymbol{v}\|_{[L^p(\Omega)]^2} \leqslant c \|\boldsymbol{v}\|_{1,2}^{1/2} \|\boldsymbol{v}\|_{0,2}^{1/2} \quad \forall \boldsymbol{v} \in H_0^1(\Omega), \qquad 1/p = \tfrac{1}{2} - \tfrac{1}{2} N^{-1} = \tfrac{1}{4} \quad (N = 2)$$

holds and

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \leqslant c \|\boldsymbol{u}\|_{1,2}^{1/2} \|\boldsymbol{u}\|_{0,2}^{1/2} \|\boldsymbol{w}\|_{1,2}^{1/2} \|\boldsymbol{w}\|_{0,2}^{1/2} \|\boldsymbol{v}\|_{1,2}.$$

For $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ it holds

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) + b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) = 0, \qquad b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{v}) = 0.$$

The main result of the paper is the following theorem.

Theorem 1. *Let assumptions* (A1)–(A6) *be satisfied for every* $t \in (0,1\rangle$. *Then scalar functions* H, Θ *and* c *and a vector function* $\boldsymbol{u}$ *exist such that*

$$H \in L^2(I; H_0^1(\Omega)) \cap L^\infty(I; L^2(\Omega)), \quad \partial_t H \in L^2(I; (H^{-1}(\Omega)),$$

$$\boldsymbol{u} \in L^2(I; V) \cap L^\infty(I; H), \quad \partial_t \boldsymbol{u} \in L^2(I; V'),$$

$$c \in L^2(I; H^1(\Omega)) \cap L^\infty(I; L^2(\Omega)), \quad \partial_t c \in L^2(I; (H^1(\Omega))')$$

and satisfying (42)–(46).

5. Proof of Theorem 1

To prove Theorem 1, the regularization of (42)–(44) will be used. The technique used will be similar to that of [23] and results of [1] will also be used. First, we define the extension of functions T and c_L on $\mathbb{R}^2$ by setting $T(x,y)=T(x^+,y^+)$ and $c_L(x,y)=c_L(x^+,y^+)$, where $r^+=\max(r,0)$. Then the regularization of T and c_L is the following:

Let $\mathscr{D}(\mathbb{R}^2)$ be the space of functions C^∞ with compact support in $\mathbb{R}^2$, let $\{R_\varepsilon\}\subset\mathscr{D}(\mathbb{R}^2)$ be a family of mollifiers[4] such that

$$\operatorname{supp}(R_\varepsilon)\subset B((0,0),\varepsilon)=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<\varepsilon^2\},\quad \varepsilon\in(0,1\rangle.$$

The regularizations of T and c_L are defined by

$$T^\varepsilon(x,y)=\varepsilon y+\int_{\mathbb{R}^2} R_\varepsilon(x+2\varepsilon-r,y-\varepsilon-s)T(r^+,s^+)\,\mathrm{d}r\,\mathrm{d}s,$$

$$c_L^\varepsilon(x,y)=\int_{\mathbb{R}^2} R_\varepsilon(x-\varepsilon-r,y-s)c_L(r^+,s^+)\,\mathrm{d}r\,\mathrm{d}s,$$

where $r^+=\max(r,0)$, $s^+=\max(s,0)$. These regularized functions Θ^ε, c_L^ε have the following properties given in the next lemma (see [1, 12]).

Lemma 3. *Let assumptions* (A1), (A2), (A4), (A5) *be satisfied. Let α be the function satisfying* (A2)(iii), (A4)(iii) *and let $\varepsilon\in(0,1\rangle$. Then the regularized functions T^ε and c_L^ε satisfy*

(i) $T^\varepsilon(x,0)=0\quad \forall x\in\mathbb{R}$,

(ii) $c_L^\varepsilon(0,y)=0\quad \forall y\in\mathbb{R}$,

(iii) *a function $C=C(\varepsilon)$ exists such that*

$$|c_L^\varepsilon(x,y)|\leqslant C|x|\quad \forall(x,y)\in\mathbb{R}\times\mathbb{R},$$

(iv) *there exists* $c_1=\text{const.}$ *satisfying* (A4)(iii) *such that*

$$|c_L^\varepsilon(x,y)|\leqslant c_1(\alpha''(x+\varepsilon))^{-1}\quad \forall(x,y)\in\mathbb{R}_+\times\mathbb{R},$$

where $\mathbb{R}_+$ is a set of non-negative reals,

---

[4] Let $\varepsilon>0$, let φ be a function satisfying $\varphi\in C_0^\infty(\mathbb{R}^2)$, $\varphi(x)\geqslant 0\ \forall x\in\mathbb{R}^2$, $\int_{\mathbb{R}^2}\varphi(x)\,\mathrm{d}x=1$, $\operatorname{supp}\varphi=\{x\in\mathbb{R}^2;|x|\leqslant 1\}$. For $u\in L^1(\Omega)$ put

$$(R_\varepsilon u)(x)=\varepsilon^{-2}\int_\Omega \varphi((x-y)\varepsilon^{-1})u(y)\,\mathrm{d}y, \tag{+}$$

i.e.

$$(R_\varepsilon u)(x)=\varepsilon^{-2}\int_{B(0,1)} u(x-\varepsilon y)\varphi(y)\,\mathrm{d}y,\ \text{where } B(0,1)=\{y\in\mathbb{R}^2;|y|<1\}.$$

The mapping R_ε defined by (+) is called mollifier. By S_ε we denote a set of all mollifiers.

(v) *there exists* $c_2 = const.$, *independent of* ε, *such that*

$$\frac{|\partial_1 T^\varepsilon(x,y)|}{\partial_2 T^\varepsilon(x,y)} \leqslant c_2 \alpha''(x+\varepsilon) \quad \forall (x,y) \in \mathbb{R}_+ \times \mathbb{R}.$$

Proof. The proof of (i)–(iv) follows from the definitions of T^ε and $c_{\mathrm{L}}^\varepsilon$ and from assumptions (A2), (A4).

To prove (i), then from the definition of $T_{\mathrm{m}}^\varepsilon$ we find

$$T^\varepsilon(x,0) = \int_{\mathbb{R}^2} R_\varepsilon(x+2\varepsilon-r, -\varepsilon-s) T(r^+, s^+)\, \mathrm{d}r\, \mathrm{d}s = 0$$

as $T(r^+,s^+)>0$ only for $s>0$ and $R_\varepsilon(x-2\varepsilon-r,-\varepsilon-s)=0$.

To prove (ii) the same technique for $c_{\mathrm{L}}^\varepsilon(0,y)$ can be used.

To prove (iii), we see that regularization $c_{\mathrm{L}}^\varepsilon$ is in class $C^1(\mathbb{R}^2)$. Since $c_{\mathrm{L}}^\varepsilon(0,y)=0$ we can write

$$c_{\mathrm{L}}^\varepsilon(x,y) = c_{\mathrm{L}}^\varepsilon(x,y) - c_{\mathrm{L}}^\varepsilon(0,y) = \int_0^x \partial_1 c_{\mathrm{L}}^\varepsilon(r,y)\, \mathrm{d}r.$$

Hence

$$|c_{\mathrm{L}}^\varepsilon(x,y)| = \left| \int_0^x \partial_1 c_{\mathrm{L}}^\varepsilon(r,y)\, \mathrm{d}r \right| \leqslant \int_0^x c \|\partial_1 c_{\mathrm{L}}^\varepsilon(r,y)\|_{L^\infty(\mathbb{R}^2)}\, \mathrm{d}r \leqslant C|x|,$$

where $|\partial_1 c_{\mathrm{L}}^\varepsilon(r,y)| \leqslant c \|\partial_1 c_{\mathrm{L}}^\varepsilon\|_{L^\infty(\mathbb{R}^2)}$ depending on ε.

To prove (iv) we start from the definition of $c_{\mathrm{L}}^\varepsilon$. The assumption (A4)(iii), applying on the integrand, was also used.

To prove (v), we assume that $\Omega_\varepsilon = B((x+2\varepsilon, y-\varepsilon),\varepsilon) \cap (\mathbb{R}_+^2 \setminus \bigcup_{i\in\mathcal{N}} O_i)$ and $(x,y)\in\mathbb{R}_+\times\mathbb{R}$, where $B((x+2\varepsilon, y-\varepsilon),\varepsilon)$ is the open ball with the centre at $(x+2\varepsilon, y-\varepsilon)$ and radius ε.

If $\mathrm{meas}(\Omega_\varepsilon)=0$ then since $\partial_1 T_\varepsilon(x,y)=0$ the assertion trivially follows. If $\mathrm{meas}(\Omega_\varepsilon)>0$ then due to definition of T^ε we have

$$\frac{|\partial_1 T^\varepsilon(x,y)|}{|\partial_2 T^\varepsilon(x,y)|} \leqslant \frac{\int_{\Omega_\varepsilon} R_\varepsilon(x+2\varepsilon-r, y-\varepsilon-s)|\partial_1 T(r,s)|\, \mathrm{d}r\, \mathrm{d}s}{\varepsilon + \int_{\Omega_\varepsilon} R_\varepsilon(x+2\varepsilon-r, y-\varepsilon-s)\partial_2 T(r,s)\, \mathrm{d}r\, \mathrm{d}s} \equiv \frac{I_{1\varepsilon}}{\varepsilon + I_{2\varepsilon}}.$$

Since, due to the definition of mollifier

$$I_\varepsilon \equiv \int_\Omega R_\varepsilon(x+2\varepsilon-r, y-\varepsilon-s)\, \mathrm{d}r\, \mathrm{d}s \leqslant 1,$$

then

$$\begin{aligned}\frac{I_{1\varepsilon}}{\varepsilon+I_{2\varepsilon}} &\leqslant \frac{I_\varepsilon \mathit{vrai}\max_{(x,y)\in\mathbb{R}_+^2} |\partial_1 T(x,y)|}{\varepsilon + I_\varepsilon \mathit{vrai}\max_{(x,y)\in\Omega_\varepsilon} \partial_2 T(x,y)} \leqslant \frac{\mathit{vrai}\max_{(x,y)\in\mathbb{R}_+^2} |\partial_1 T(x,y)|}{\varepsilon + (\alpha''(x+\varepsilon))^{-1}} \\ &\leqslant \alpha''(x+\varepsilon) \mathit{vrai} \max_{(x,y)\in\mathbb{R}_+^2} |\partial_1 T(x,y)| \leqslant c_2\alpha''(x+\varepsilon),\end{aligned}$$

which completes the proof. □

Since $j(\boldsymbol{v}(t)) = 2\int_\Omega (D_{\rm II}(\boldsymbol{v}))^{1/2}\,\mathrm{d}\boldsymbol{x}$ is the nondifferentiable functional and since $J(\boldsymbol{v}) = \int_\Omega \hat{g}j(\boldsymbol{v}(t))\,\mathrm{d}t$ then $J_\varepsilon(\boldsymbol{v})$ denotes a regularized functional $J(\boldsymbol{v})$, defined by

$$J_\varepsilon(\boldsymbol{v}) = \int_I \hat{g}j_\varepsilon(\boldsymbol{v}(t))\,\mathrm{d}t,$$

where

$$j_\varepsilon(\boldsymbol{v}(t)) = \frac{2}{1+\varepsilon}\int_\Omega (D_{\rm II}(\boldsymbol{v}(t))^{(1/2)(1+\varepsilon)}\,\mathrm{d}\boldsymbol{x},$$

$\varepsilon > 0$, for which

$$(J'_\varepsilon(\boldsymbol{v}), \boldsymbol{w}) = \int_{\Omega\times I} \hat{g}(D_{\rm II}(\boldsymbol{v}))^{1/2(\varepsilon-1)} D_{ij}(\boldsymbol{v}) D_{ij}(\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Then the weak regularized solution to Problem $(\mathscr{P}_{\rm centh})_v$ can be defined. The idea is to approximate the variational inequality by the variational equality.

Let assumptions (A1)–(A5) be satisfied for every $\varepsilon \in (0,1\rangle$. Let $H^\varepsilon \in L^2(I;{}^1V)\cap H^1(I;H^{-1}(\Omega))$, $\boldsymbol{u}^\varepsilon \in L^\infty(I;H)\cap L^2(I;V)$, $c^\varepsilon \in L^2(I;{}^2V)\cap H^1(I;(H^1(\Omega))')$. Then the tetrad $\{H^\varepsilon, \Theta^\varepsilon, \boldsymbol{u}^\varepsilon, c^\varepsilon\}$ is a weak solution to the regularized problem to Problem $(\mathscr{P}_{\rm centh})_v$ if the following hold:

$$\int_I \{(\partial_t H^\varepsilon(t), z - \Theta^\varepsilon(t)) + b_\Theta(\boldsymbol{u}^\varepsilon(t), \Theta^\varepsilon(t), z - \Theta^\varepsilon(t)) + a_\Theta(\Theta^\varepsilon(t), z - \Theta^\varepsilon(t))$$
$$+ b_p(\boldsymbol{u}^\varepsilon(t), z - \Theta^\varepsilon(t)) - (Q(t), z - \Theta^\varepsilon(t))\}\,\mathrm{d}t \geqslant 0 \quad \forall z \in {}^1W, \tag{47}$$

$$\int_I \{(\partial_t \boldsymbol{u}^\varepsilon(t), \boldsymbol{v} - \boldsymbol{u}^\varepsilon(t)) + \hat{\mu}a(\boldsymbol{u}^\varepsilon(t), \boldsymbol{v} - \boldsymbol{u}^\varepsilon(t)) + b(\boldsymbol{u}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t), \boldsymbol{v} - \boldsymbol{u}^\varepsilon(t))$$
$$+ \hat{g}j_\varepsilon(\boldsymbol{v}) - \hat{g}j_\varepsilon(\boldsymbol{u}^\varepsilon(t)) + b_s(\Theta^\varepsilon(t) - \Theta_0, \boldsymbol{v} - \boldsymbol{u}^\varepsilon(t)) - (\boldsymbol{F}(t), \boldsymbol{v} - \boldsymbol{u}^\varepsilon(t))\}\,\mathrm{d}t \geqslant 0 \quad \forall \boldsymbol{v} \in W, \tag{48}$$

$$\int_I \{(\partial_t c^\varepsilon(t), d - c^\varepsilon(t)) + b_c(\boldsymbol{u}^\varepsilon(t), c^\varepsilon_{\rm L}(t), d - c^\varepsilon(t)) + a_c(c^\varepsilon(t), d - c^\varepsilon(t))$$
$$- (\varphi(t), d - c^\varepsilon(t))\}\,\mathrm{d}t \geqslant 0, \quad \forall d \in {}^2W, \tag{49}$$

$$H^\varepsilon(\boldsymbol{x}, t_0) = H_0(\boldsymbol{x}), \qquad \boldsymbol{u}^\varepsilon(\boldsymbol{x}, t_0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad c^\varepsilon(\boldsymbol{x}, t_0) = c_0(\boldsymbol{x}) \quad \text{a.e. in } \Omega, \tag{50}$$

where $\Theta^\varepsilon = T^\varepsilon(c^\varepsilon, H^\varepsilon)$.

The method of proof is the following:

(a) The existence of the solution of (47)–(50) based on the Galerkin approximation will be proved.
(b) A priori estimates I and II independent of ε will be derived.
(c) Limitation processes over m and ε.
(d) Limitation process $\hat{g} \to 0$ and the existence of multipliers.

The existence of $\{H^\varepsilon, \Theta^\varepsilon, \boldsymbol{u}^\varepsilon, c^\varepsilon\}$ will be proved by means of the finite-dimensional approximation. Let $(\xi_j\}$, $\{\zeta_j\}$ be two orthogonal bases of the space $L^2(\Omega)$, composed of eigenfunctions of the operator $-\Delta$ over the domain Ω, where the first is relative to the Dirichlet homogeneous boundary conditions and the second one to the Neumann homogeneous boundary conditions, and $\{\boldsymbol{w}_j\}$ be a countable basis of the space V composed of eigenfunctions of the canonical isomorphism Λ of

$V \to V'$, i.e., $((\boldsymbol{w}_j, \boldsymbol{v})) = \lambda_j(\boldsymbol{w}_j, \boldsymbol{v})\ \forall \boldsymbol{v} \in V$, $|\boldsymbol{w}_j| = 1$, i.e., each finite subsets of bases $\{\xi_j\}$, $\{\zeta_j\}$ and $\{\boldsymbol{w}_j\}$ are linearly independent and $\operatorname{span}\{\xi_j \mid j = 1, 2, \ldots\}$, $\operatorname{span}\{\boldsymbol{w}_j \mid j = 1, 2, \ldots\}$ and $\operatorname{span}\{\zeta_j \mid j = 1, 2, \ldots\}$ are dense in $H_0^1(\Omega)$, or in V, or in $H^1(\Omega)$, respectively, as $H_0^1(\Omega)$, V, $H^1(\Omega)$ are separable spaces. Let ${}^1V^m$, V^m, ${}^2V^m$ be spaces spanned by $\{\xi_j \mid 1 \leqslant j \leqslant m\}$, $\{\boldsymbol{w}_j \mid 1 \leqslant j \leqslant m\}$ and $\{\zeta_j \mid 1 \leqslant j \leqslant m\}$, respectively. Then the approximations $H_m^\varepsilon \in {}^1V^m$, $\boldsymbol{u}_m^\varepsilon \in V^m$, $c_m^\varepsilon \in {}^2V^m$ of the order m satisfy

$$(\partial_t H_m^\varepsilon(t), z_j) + b_\Theta(\boldsymbol{u}_m^\varepsilon(t), \Theta_m^\varepsilon(t), z_j) + a_\Theta(\Theta_m^\varepsilon(t), z_j) + b_p(\boldsymbol{u}_m^\varepsilon(t), z_j)$$
$$- (Q(t), z_j) = 0, \quad 1 \leqslant j \leqslant m, \tag{51}$$

$$(\partial_t \boldsymbol{u}_m^\varepsilon(t), \boldsymbol{v}_j) + \hat{\mu} a(\boldsymbol{u}_m^\varepsilon(t), \boldsymbol{v}_j) + b(\boldsymbol{u}_m^\varepsilon(t), \boldsymbol{u}_m^\varepsilon(t), \boldsymbol{v}_j) + \hat{g}(j_\varepsilon'(\boldsymbol{u}_m^\varepsilon(t)), \boldsymbol{v}_j)$$
$$+ b_s(\Theta_m^\varepsilon(t) - \Theta_0, \boldsymbol{v}_j) - (\boldsymbol{F}(t), \boldsymbol{v}_j) = 0, \quad 1 \leqslant j \leqslant m, \tag{52}$$

$$(\partial_t c_m^\varepsilon(t), d_j) + b_c(\boldsymbol{u}_m^\varepsilon(t), c_m^\varepsilon(t), d_j) + a_c(c_m^\varepsilon(t), d_j) - (\varphi(t), d_j) = 0, \quad 1 \leqslant j \leqslant m, \tag{53}$$

verified for almost every $t \in I$, for every $z \in {}^1V^m$, $\boldsymbol{v} \in V^m$, $d \in {}^2V^m$ and the initial conditions

$$H_m^\varepsilon(\boldsymbol{x}, t_0) = \sum_{i=1}^m (H_0, \xi_i)\xi_i(\boldsymbol{x}), \qquad \boldsymbol{u}_m^\varepsilon(\boldsymbol{x}, t_0) = \sum_{i=1}^m (\boldsymbol{u}_0, \boldsymbol{w}_i)\boldsymbol{w}_i(\boldsymbol{x}),$$

$$c_m^\varepsilon(\boldsymbol{x}, t_0) = \sum_{i=1}^m (c_0, \zeta_i)\zeta_i(\boldsymbol{x}), \tag{54}$$

verified for a.e. $\boldsymbol{x} \in \Omega$. Since $\{\xi_j\}_{j=1}^m$, $\{\boldsymbol{w}_j\}_{j=1}^m$, $\{\zeta_j\}_{j=1}^m$ are linearly independent, the system (51)–(54) is a regular system of ordinary differential equations of the first order and therefore (51)–(54) uniquely define $\{H_m^\varepsilon, \Theta_m^\varepsilon, \boldsymbol{u}_m^\varepsilon, c_m^\varepsilon\}$ on the interval $I_m = \langle t_0, t_m \rangle$. Therefore, (51)–(54) is valid for every test function

$$z(t) = \sum_{i=1}^m a_i(t)\xi_i, \qquad \boldsymbol{v}(t) = \sum_{i=1}^m b_i(t)\boldsymbol{w}_i, \qquad d(t) = \sum_{i=1}^m \gamma_i(t)d_i, \quad t \in I_m,$$

where a_i, b_i, γ_i are continuously differentiable functions on I_m, $i = 1, \ldots, m$. Particularly, it holds for $z(t) = H_m^\varepsilon(t)$, $v(t) = \boldsymbol{u}_m^\varepsilon(t)$, $d(t) = c_m^\varepsilon(t)$, $t \in I_m$.

Now we discuss a priori estimates I and II.

5.1. *A priori estimates* I

Let us introduce the notation

$$X_m = b_s(\Theta_m^\varepsilon - \Theta_0, \boldsymbol{u}_m^\varepsilon) + b_p(\boldsymbol{u}_m^\varepsilon, \Theta_m^\varepsilon).$$

Let $\partial\beta_{ij}/\partial x_j \equiv \partial_j\beta_{ij} \in L^\infty(\Omega)\ \forall i, j$. Then a positive constant c, independent of m and ε exists such that

$$|X_m| = |b_s(\Theta_m^\varepsilon - \Theta_0, \boldsymbol{u}_m^\varepsilon) + b_p(\boldsymbol{u}_m^\varepsilon, \Theta_m^\varepsilon)|$$
$$\leqslant c(1 + \|\Theta_m^\varepsilon(t)\|_{1,1}\|\boldsymbol{u}_m^\varepsilon(t)\|_{0,2} + \|\Theta_m^\varepsilon(t)\|_{0,1}\|\boldsymbol{u}_m^\varepsilon(t)\|_{1,2}). \tag{55}$$

In the next section we shall denote all used constants by type c without any indices (if possible). Via integration of (54) with $z=H_m^\varepsilon(t)$, $\boldsymbol{v}(t)=\boldsymbol{u}_m^\varepsilon(t)$, $d(t)=c_m^\varepsilon(t)$ in time t over $I_m=(t_0,t_m)$, we obtain

$$\begin{aligned}\int_{I_m}\{&(\partial_t H_m^\varepsilon(t),H_m^\varepsilon(t))+(\partial_t \boldsymbol{u}_m^\varepsilon(t),\boldsymbol{u}_m^\varepsilon(t))+(\partial_t c_m^\varepsilon(t),c_m^\varepsilon(t))\\ &+a_\Theta(\Theta_m^\varepsilon(t),H_m^\varepsilon(t))+\hat{\mu}a(\boldsymbol{u}_m^\varepsilon(t),\boldsymbol{u}_m^\varepsilon(t))+a_c(c_m^\varepsilon(t),c_m^\varepsilon(t))\\ &+b_\Theta(\boldsymbol{u}_m^\varepsilon(t),\Theta_m^\varepsilon(t),H_m^\varepsilon(t))+b(\boldsymbol{u}_m^\varepsilon(t),\boldsymbol{u}_m^\varepsilon(t),\boldsymbol{u}_m^\varepsilon(t))\\ &+b_c(\boldsymbol{u}_m^\varepsilon(t),c_m^\varepsilon(t),c_m^\varepsilon(t))+b_s(\Theta_m^\varepsilon(t)-\Theta_0,\boldsymbol{u}_m^\varepsilon(t))\\ &+b_p(\boldsymbol{u}_m^\varepsilon(t),\Theta_m^\varepsilon(t))+\hat{g}(j'(\boldsymbol{u}_m^\varepsilon(t)),\boldsymbol{u}_m^\varepsilon(t))-(Q(t),H_m^\varepsilon(t))\\ &-(\boldsymbol{F}(t),\boldsymbol{u}_m^\varepsilon(t))-(\varphi(t),c_m^\varepsilon(t))\}\,\mathrm{d}t=0.\end{aligned}$$

We find

$$b(\boldsymbol{u}_m^\varepsilon(t),\boldsymbol{u}_m^\varepsilon(t),\boldsymbol{u}_m^\varepsilon(t))=0,$$

$$(J'(\boldsymbol{u}_m^\varepsilon(t)),\boldsymbol{u}_m^\varepsilon(t))=\int_{I_m}\hat{g}(j_\varepsilon'(\boldsymbol{u}_m^\varepsilon(t)),\boldsymbol{u}_m^\varepsilon(t))\,\mathrm{d}t\geqslant 0,\quad \text{as}\quad (j_\varepsilon'(\boldsymbol{v}),\boldsymbol{v})\geqslant 0.$$

$$\int_{I_m}(\partial_t H_m^\varepsilon(t),H_m^\varepsilon(t))\,\mathrm{d}t=\frac{1}{2}\int_\Omega(H_m^\varepsilon(t))^2\,\mathrm{d}\boldsymbol{x}-\frac{1}{2}\int_\Omega(H_m^\varepsilon(t_0))^2\,\mathrm{d}\boldsymbol{x}$$

and similarly for remaining terms.

Since $\Theta^\varepsilon=T^\varepsilon(c_m^\varepsilon,H_m^\varepsilon)$, then using Green's theorem, after modification grad T^ε by the chain rule, using (A2)(i)(ii), $\partial_2 T^\varepsilon\geqslant\varepsilon$ and resulting estimate $|T^\varepsilon(x,y)|\leqslant\|\partial_2 T^\varepsilon\|_{L^\infty(\mathbb{R}^2)}|y|$, we obtain

$$\begin{aligned}&\int_{I_m}(a_\Theta(\Theta_m^\varepsilon(t),H_m^\varepsilon(t))+b_\Theta(\boldsymbol{u}_m^\varepsilon(t),\Theta_m^\varepsilon(t),H_m^\varepsilon(t))\,\mathrm{d}t\\ &\quad=\int_{I_m}\int_\Omega \operatorname{grad} T^\varepsilon(c_m^\varepsilon(t),H_m^\varepsilon(t))\operatorname{grad} H_m^\varepsilon(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &\qquad+\int_{I_m}\int_\Omega \boldsymbol{u}_m^\varepsilon(t)\operatorname{grad} T^\varepsilon(c_m^\varepsilon(t),H_m^\varepsilon(t))H_m^\varepsilon(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &\quad=\int_{I_m}\int_\Omega \operatorname{grad} T^\varepsilon(c_m^\varepsilon(t),H_m^\varepsilon(t))\operatorname{grad} H_m^\varepsilon(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &\qquad-\int_{I_m}\int_\Omega T^\varepsilon(c_m^\varepsilon(t),H_m^\varepsilon(t))\boldsymbol{u}_m^\varepsilon(t)\operatorname{grad} H_m^\varepsilon(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &\quad\leqslant\varepsilon\int_{I_m}\int_\Omega\|\operatorname{grad} H_m^\varepsilon(t)\|_{1,1}^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &\qquad-\|\partial_2 T^\varepsilon(t)\|_{L^\infty(\mathbb{R}^2)}\int_{I_m}\int_\Omega|H_m^\varepsilon(t)|\;|\boldsymbol{u}_m^\varepsilon(t)\operatorname{grad} H_m^\varepsilon(t)|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &\qquad-\int_{I_m}\int_\Omega|\partial_1 T^\varepsilon(c_m^\varepsilon(t),H_m^\varepsilon(t))|\;|\operatorname{grad} c_m^\varepsilon(t)\operatorname{grad} H_m^\varepsilon(t)|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned}$$

Likewise we estimate

$$\int_{I_m} (a_c(c_m^\varepsilon(t), c_m^\varepsilon(t)) + b_c(\boldsymbol{u}_m^\varepsilon(t), c_m^\varepsilon(t), c_m^\varepsilon(t))\,\mathrm{d}t.$$

To estimate

$$\|\partial_2 T^\varepsilon(t)\|_{L^\infty(\mathbb{R}^2)} \int_{I_m}\int_\Omega |H_m^\varepsilon(t)|\;|\boldsymbol{u}_m^\varepsilon(t)\,\mathrm{grad}\,H_m^\varepsilon(t)|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$\leqslant \|\partial_2 T^\varepsilon(t)\|_{L^\infty(\mathbb{R}^2)} \int_{I_m}\int_\Omega |H_m^\varepsilon(t)|\;\|\boldsymbol{u}_m^\varepsilon(t)\|\;\|\mathrm{grad}\,H_m^\varepsilon(t)\|\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$\leqslant \gamma_0\|\boldsymbol{u}_m^\varepsilon(t)\|_{L^\infty(\Omega\times I)} \int_{I_m}\int_\Omega |H_m^\varepsilon(t)|\;\|\,\mathrm{grad}\,H_m^\varepsilon(t)\|\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$\leqslant \gamma_1^2\varepsilon^{-1}\int_{I_m}\int_\Omega (H_m^\varepsilon(t))^2\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \frac{1}{4}\varepsilon\int_{I_m}\int_\Omega \|\mathrm{grad}\,H_m^\varepsilon(t)\|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

where $\gamma_1 = \gamma_0\|\boldsymbol{u}_m^\varepsilon(t)\|_{L^\infty(\Omega\times I)}$, where γ_0 is the Lipschitz constant and the Young's inequality [5] was applied. Similarly,

$$\int_{I_m}\int_\Omega |\partial_1 T^\varepsilon(c_m^\varepsilon(t), H_m^\varepsilon(t))|\;|\mathrm{grad}\,c_m^\varepsilon(t)\,\mathrm{grad}\,H_m^\varepsilon(t)|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$\leqslant \|\partial_1 T^\varepsilon(c_m^\varepsilon(t), H_m^\varepsilon(t))\|_{L^\infty(\mathbb{R}^2)} \int_{I_m}\int_\Omega \|\mathrm{grad}\,c_m^\varepsilon(t)\|\;\|\mathrm{grad}\,H_m^\varepsilon(t)\|\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$\leqslant \gamma_1^2\varepsilon^{-1}\int_{I_m}\int_\Omega \|\,\mathrm{grad}\,c_m^\varepsilon(t)\|\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \frac{1}{4}\varepsilon\int_{I_m}\int_\Omega \|\mathrm{grad}\;H_m^\varepsilon(t)\|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Moreover,

$$\int_{I_m} (\boldsymbol{F}(t), \boldsymbol{u}_m^\varepsilon(t))\,\mathrm{d}t \leqslant \int_{I_m} [\|\boldsymbol{f}(t)\|_*\|\boldsymbol{u}_m^\varepsilon(t)\|_{1,2} + \|\boldsymbol{P}_0(t)\|_*\|\boldsymbol{u}_m^\varepsilon(t)\|_{1,2}]\,\mathrm{d}t,$$

where by $\|\boldsymbol{f}\|_*$ we denote the norm in V', [6] and furthermore,

$$2\int_{I_m} \|\boldsymbol{F}(t)\|_*\|\boldsymbol{u}_m^\varepsilon(t)\|_{1,2}\,\mathrm{d}t \leqslant c\hat{\mu}\int_{I_m} \|\boldsymbol{u}_m^\varepsilon(t)\|_{1,2}^2\,\mathrm{d}t + (c\hat{\mu})^{-1}\int_{I_m} \|\boldsymbol{F}(t)\|_*^2\,\mathrm{d}t.$$

---

[5] We say that Φ is a Young's function if $\Phi(t) = \int_0^t \varphi(\tau)\,\mathrm{d}\tau$, $t \geqslant 0$, where the real-valued function $\varphi \in \langle 0, \infty)$ is of properties: (i) $\varphi(0) = 0$, (ii) $\varphi(\tau) > 0$ for $\tau > 0$, (iii) φ is right-continuous at any point $\tau \geqslant 0$, (iv) φ is a nondecreasing function on $\langle 0, \infty)$, (v) $\lim \varphi(\infty) = \infty$. Moreover, we say that function Ψ is the complementary function to Φ if $\Psi(t) = \int_0^t \psi(\tau)\,\mathrm{d}\tau$, where $\psi(t) = \sup_{\varphi(\tau)\leqslant t} \tau$, $t \geqslant 0$.

Young's inequality: Let Φ, Ψ be a pair of complementary Young's functions. Then for all functions $u, v \in \langle 0, \infty)$ $uv \leqslant \Phi(u) + \Phi(v)$ and equality holds if and only if $v = \varphi(u)$ or $u = \psi(v)$.

[6] We define it as dual to $\|\boldsymbol{f}\| = \sup_{\boldsymbol{v}\in V} |(\boldsymbol{f}, \boldsymbol{v})|\|^{-1}$.

Similarly, we estimate $\int_{I_m}(Q(t), H_m^{\varepsilon}(t))\,dt$ and $\int_{I_m}(\varphi(t), c_m^{\varepsilon}(t))\,dt$. Then, summing all above obtained estimates, using (55) and the estimates of bilinear and trilinear forms, applying the Gronwall's lemma, after some modifications we find the estimates

$$\|H_m^{\varepsilon}(t)\|_{0,1} \leqslant c_0(\varepsilon), \quad t \in I_m, \qquad \int_{I_m} \|H_m^{\varepsilon}(t)\|_{1,1}\,dt \leqslant c_0(\varepsilon), \tag{56}$$

$$\|\boldsymbol{u}_m^{\varepsilon}(t)\|_{0,2} \leqslant c, \quad t \in I_m, \qquad \int_{I_m} \|\boldsymbol{u}_m^{\varepsilon}(t)\|_{1,2}^2\,dt \leqslant c,$$

$$\|c_m^{\varepsilon}(t)\|_{0,1} \leqslant c, \quad t \in I_m, \qquad \int_{I_m} \|c_m^{\varepsilon}(t)\|_{1,1}\,dt \leqslant c, \quad \varepsilon \in (0,1\rangle,$$

where constants c are independent of m and ε and c_0 is independent of m.

From these estimates we obtain

$$\begin{aligned}
&\{H_m^{\varepsilon}(t),\ m \in \mathcal{N},\ \varepsilon \in (0,1\rangle\} && \text{is a bounded subset in } L^2(I; {}^1V) \cap L^{\infty}(I; L^2(\Omega)),\\
&\{\boldsymbol{u}_m^{\varepsilon}(t),\ m \in \mathcal{N},\ \varepsilon \in (0,1\rangle\} && \text{is a bounded subset in } L^2(I; V) \cap L^{\infty}(I; H(\Omega)),\\
&\{c_m^{\varepsilon}(t),\ m \in \mathcal{N},\ \varepsilon \in (0,1\rangle\} && \text{is a bounded subset in } L^2(I; {}^2V) \cap L^{\infty}(I; L^2(\Omega)).
\end{aligned} \tag{57}$$

5.2. *A priori estimates* II

Now we shall show that

$$\begin{aligned}
&\{\partial_t H_m^{\varepsilon}(t),\ m \in \mathcal{N},\ \varepsilon \in (0,1\rangle\} && \text{is a bounded subset in } L^2(I; H^{-1}(\Omega)),\\
&\{\partial_t \boldsymbol{u}_m^{\varepsilon}(t),\ m \in \mathcal{N},\ \varepsilon \in (0,1\rangle\} && \text{is a bounded subset in } L^2(I; V'),\\
&\{\partial_t c_m^{\varepsilon}(t),\ m \in \mathcal{N},\ \varepsilon \in (0,1\rangle\} && \text{is a bounded subset in } L^2(I; (H^1(\Omega))'),
\end{aligned} \tag{58}$$

independent of m and ε.

By virtue of estimates given above

$$|b(\boldsymbol{u}_m^{\varepsilon}(t), \boldsymbol{u}_m^{\varepsilon}(t), \boldsymbol{v})| \leqslant c\|\boldsymbol{u}_m^{\varepsilon}(t)\|_{1,2}\|\boldsymbol{u}_m^{\varepsilon}(t)\|_{0,2}\|\boldsymbol{v}\|_{1,2}$$

and since due to (56)

$$|b(\boldsymbol{u}_m^{\varepsilon}(t), \boldsymbol{u}_m^{\varepsilon}(t), \boldsymbol{v})| \leqslant c\|\boldsymbol{u}_m^{\varepsilon}(t)\|_{1,2}\|\boldsymbol{v}\|_{1,2},$$

then

$$b(\boldsymbol{u}_m^{\varepsilon}(t), \boldsymbol{u}_m^{\varepsilon}(t), \boldsymbol{v}) = (h_{um}(t), \boldsymbol{v}), \quad \forall \boldsymbol{v} \in V,$$

and similarly,

$$b_{\Theta}(\boldsymbol{u}_m^{\varepsilon}(t), \Theta_m^{\varepsilon}(t), z) = (h_{\Theta m}(t), \boldsymbol{v}), \quad \forall z \in {}^1V,$$

$$b_c(\boldsymbol{u}_m^{\varepsilon}(t), c_m^{\varepsilon}(t), d) = (h_{cm}(t), d), \quad \forall d \in {}^2V,$$

$h_{um}, h_{\Theta m}, h_{cm}$ remaining in bounded sets of $L^2(I; V')$ or $L^2(I; {}^1V')$ or $L^2(I; {}^2V')$, respectively. The coupled terms are from $L^2(\Omega)$ or $[L^2(\Omega)]^2$, respectively, and therefore are included into Q and $\boldsymbol{F}$.

Linear forms $\Theta \to a_\Theta(\Theta,z)$ for a fixed $z \in {}^1V$ is continuous on ${}^1V$ so that

$$a_\Theta(\Theta(t),z)=(A_\Theta\Theta(t),z), \qquad A_\Theta \in \mathscr{L}({}^1V,\ {}^1V'). \tag{59}$$

Similarly, forms $\boldsymbol{u} \to a(\boldsymbol{u},\boldsymbol{v})$ and $c \to a_c(c,w)$ for fixed $\boldsymbol{u} \in V$ or $c \in {}^2V$, respectively, are continuous on V or ${}^2V$, respectively, so that

$$a(\boldsymbol{u}(t),\boldsymbol{v})=(A_u\boldsymbol{u}(t),\boldsymbol{v}), \qquad A_u \in \mathscr{L}(V,V'). \tag{60}$$

$$a_c(c(t),d)=(A_c c(t),d), \qquad A_c \in \mathscr{L}({}^2V,{}^2V'). \tag{61}$$

Then (51)–(54) are equivalent to

$$\begin{aligned}
&(\partial_t H_m^\varepsilon + A_\Theta\Theta_m^\varepsilon + h_{\Theta m} - Q, z_j)=0, \quad 1 \leqslant j \leqslant m,\\
&(\partial_t \boldsymbol{u}_m^\varepsilon + \hat{\mu}A_u\boldsymbol{u}_m^\varepsilon + h_{um} + \hat{g}j_\varepsilon'(\boldsymbol{u}_m^\varepsilon) - \boldsymbol{F}, \boldsymbol{v}_j)=0, \quad 1 \leqslant j \leqslant m,\\
&(\partial_t c_m^\varepsilon + A_c c_m^\varepsilon + h_{cm} - \varphi, d_j)=0, \quad 1 \leqslant j \leqslant m.
\end{aligned} \tag{62}$$

Let
$S_{\Theta m}$ be orthogonal projection $L^2(\Omega) \to W_\Theta^m = \operatorname{span}\{z_j \mid 1 \leqslant j \leqslant m\}$,
S_{um} be orthogonal projection $H \to W^m = \operatorname{span}\{\boldsymbol{v}_j \mid 1 \leqslant j \leqslant m\}$,
S_{cm} be orthogonal projection $L^2(\Omega) \to W_c^m = \operatorname{span}\{d_j \mid 1 \leqslant j \leqslant m\}$,
then

$$S_{\Theta m}h_\Theta = \sum_{j=1}^m (h_\Theta,z_j)z_j, \quad S_{um}\boldsymbol{h}_u = \sum_{j=1}^m (\boldsymbol{h}_u,\boldsymbol{v}_j)\boldsymbol{v}_j, \quad S_{cm}h_c = \sum_{j=1}^m (h_c,d_j)d_j,$$

where $\{z_j\}$, $\{\boldsymbol{v}_j\}$, $\{d_j\}$ are orthogonal bases of W_Θ^m, or W^m, or W_c^m, respectively. Then from (62) and from the facts that $S_{\Theta m}\,\partial_t H_m^\varepsilon = \partial_t H_m^\varepsilon$, $S_{um}\,\partial_t\boldsymbol{u}_m^\varepsilon = \partial_t\boldsymbol{u}_m^\varepsilon$, $S_{cm}\,\partial_t c_m^\varepsilon = \partial_t c_m^\varepsilon$, we obtain

$$\begin{aligned}
\partial_t H_m^\varepsilon &= S_{\Theta m}(Q - A_\Theta\Theta_m^\varepsilon - h_{\Theta m}),\\
\partial_t \boldsymbol{u}_m^\varepsilon &= S_{um}(\boldsymbol{F} - \hat{\mu}A_u\boldsymbol{u}_m^\varepsilon - \hat{g}j_\varepsilon'(\boldsymbol{u}_m^\varepsilon) - h_{um}),\\
\partial_t c_m^\varepsilon &= S_{cm}(\varphi - A_c c_m^\varepsilon - h_{cm}).
\end{aligned} \tag{63}$$

Due to (56) and (57) $A_\Theta\Theta_m^\varepsilon$, $A_u\boldsymbol{u}_m^\varepsilon$, $A_c c_m^\varepsilon$ are bounded subsets of $L^2(I;{}^1V')$ or $L^2(I;V')$ or $L^2(I;{}^2V')$, respectively.

Due to the definition

$$(J_\varepsilon'(\boldsymbol{v}),\boldsymbol{w}) = \int_{\Omega\times I} \hat{g}D_{\mathrm{II}}(\boldsymbol{v})^{(1/2)(\varepsilon-1)}D_{ij}(\boldsymbol{v})D_{ij}(\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

$$\|j_\varepsilon'(\boldsymbol{v})\|_* \leqslant c\left(\int_\Omega (D_{\mathrm{II}}(\boldsymbol{v}))^\varepsilon\,\mathrm{d}\boldsymbol{x}\right)^{1/2},$$

where $\|\cdot\|_*$ denotes the dual norm, then $j_\varepsilon'(\boldsymbol{u}_m^\varepsilon)$ is a bounded subset of $L^2(I;V')$. Thus (63) indicate that $\partial_t H_m^\varepsilon = S_{\Theta m}p_{\Theta m}$, where $p_{\Theta m} \in \mathscr{P}_{\Theta p} \subset L^2(I;H^{-1}(\Omega))$, $P_{\Theta p}$ is a bounded subset of $L^2(I;H^{-1}(\Omega))$. Similarly, $\partial_t\boldsymbol{u}_m^\varepsilon = S_{um}\boldsymbol{p}_{um}$, where $\boldsymbol{p}_{um} \in \mathscr{P}_{up} \subset L^2(I;V')$, $\mathscr{P}_{up}$ is a bounded subset of $L^2(I;V')$ and finally, $\partial_t c_m^\varepsilon = S_{cm}p_{cm}$, where $p_{cm} \in \mathscr{P}_{cp} \subset L^2(I;L^2(\Omega))$, $\mathscr{P}_{cp}$ is a bounded subset of $L^2(I;(H^1(\Omega))')$.

Moreover, due to properties of eigenfunctions, then $\|S_{\Theta m} p_{\Theta m}\|_{1_{V'}} \leqslant c\|p_{\Theta m}\|_{1_{V'}}$; $\|S_{um}\boldsymbol{p}_{um}\|_{V'} \leqslant c\|\boldsymbol{p}_{um}\|_{V'}$; $\|S_{cm}\boldsymbol{p}_{cm}\|_{2_{V'}} \leqslant c\|\boldsymbol{p}_{cm}\|_{2_{V'}}$, which completes this part of the proof.

5.3. Passages to the limit over m

We shall prove the convergence of the finite-dimensional approximation for ε being fixed.

From the a priori estimates I and II as well as from (57), (58) the subsequences $\{H_\mu^\varepsilon(t), \mu \in \mathcal{N}\}$, $\{\boldsymbol{u}_\mu^\varepsilon(t), \mu \in \mathcal{N}\}$, $\{c_\mu^\varepsilon(t), \mu \in \mathcal{N}\}$ of the sequences $\{H_m^\varepsilon(t), m \in \mathcal{N}\}$, $\{\boldsymbol{u}_m^\varepsilon(t), m \in \mathcal{N}\}$ and $\{c_m^\varepsilon(t), m \in \mathcal{N}\}$, respectively, can be taken such that

$$\begin{array}{ll}
H_\mu^\varepsilon \to H^\varepsilon & \text{in } L^2(I; L^2(\Omega)) \text{ strongly},\\
H_\mu^\varepsilon \to H^\varepsilon & \text{in } L^2(I; H_0^1(\Omega)) \text{ weakly},\\
H_\mu^\varepsilon \to H^\varepsilon & \text{in } H^1(I; H^{-1}(\Omega)) \text{ weakly},\\
\boldsymbol{u}_\mu^\varepsilon \to \boldsymbol{u}^\varepsilon & \text{in } L^\infty(I; H) \ *\text{-weakly (weakly star)},\\
\boldsymbol{u}_\mu^\varepsilon \to \boldsymbol{u}^\varepsilon & \text{in } L^2(I; V) \text{ weakly},\\
\boldsymbol{u}_\mu^\varepsilon \to \boldsymbol{u}^\varepsilon & \text{in } L^2(I; H) \text{ strongly},\\
\partial_t \boldsymbol{u}_\mu^\varepsilon \to \partial_t \boldsymbol{u}^\varepsilon & \text{in } L^2(I; V') \text{ weakly},\\
c_\mu^\varepsilon \to c^\varepsilon & \text{in } L^2(I; L^2(\Omega)) \text{ strongly},\\
c_\mu^\varepsilon \to c^\varepsilon & \text{in } L^2(I; H^1(\Omega)) \text{ weakly},\\
c_\mu^\varepsilon \to c^\varepsilon & \text{in } L^2(I; (H^1(\Omega))') \text{ weakly},
\end{array} \tag{64}$$

where $H^\varepsilon \in L^2(I; H_0^1(\Omega)) \bigcap H^1(I; H^{-1}(\Omega))$, $\boldsymbol{u}^\varepsilon \in L^2(I; V) \bigcap L^\infty(I; H)$, $c^\varepsilon \in L^2(I; H^1(\Omega)) \bigcap H^1(I; (H(\Omega))')$.

Remark. The function $f_j \to f$ $*$-weakly in $L^\infty(I; [L^2(\Omega)]^N)$ if $\int_{t_0}^{t_1} (f_j(t), \varphi(t))\,\mathrm{d}t \to \int_{t_0}^{t_1} (f(t), \varphi(t))\,\mathrm{d}t$ weakly $\forall \varphi \in L^1(I; [L^2(\Omega)]^N)$.

For the appropriate components $u_{i\mu}^\varepsilon$ of $\boldsymbol{u}_\mu^\varepsilon$ we have

$$u_{i\mu}^\varepsilon \to u_i^\varepsilon \quad \text{weakly a.e. in } \Omega \times I, \tag{65}$$

since $\boldsymbol{u}_\mu^\varepsilon \to \boldsymbol{u}^\varepsilon$ in $L^2(I; H)$ strongly. Furthermore, $\{j'(\boldsymbol{u}_m^\varepsilon)\}$, $\{u_{i\mu}^\varepsilon u_{j\mu}^\varepsilon\}$, due to estimate $\|\boldsymbol{u}_\mu^\varepsilon\|_{[L^p(\Omega)]^2} \leqslant c\|\boldsymbol{u}_\mu^\varepsilon\|_{1,2}^{1/2} \|\boldsymbol{u}_\mu^\varepsilon\|_{[L^2(\Omega)]^2}^{1/2}$ $\forall \boldsymbol{u} \in [H_0^1(\Omega)]^2$, and (5.1b), are bounded subsets of spaces $L^2(I; V')$ and $L^2(I; [L^{p/2}(\Omega)]^2)$, respectively. Then we also can assume that

$$\begin{array}{ll}
j_\varepsilon'(\boldsymbol{u}_\mu^\varepsilon) \to \chi_0 & \text{weakly in } L^2(I; V'),\\
u_{i\mu}^\varepsilon u_{j\mu}^\varepsilon \to \Xi_{ij} & \text{weakly in } L^2(I; L^{p/2}(\Omega)).
\end{array} \tag{66}$$

From (65), $u_{i\mu}^\varepsilon u_{j\mu}^\varepsilon \to u_i^\varepsilon u_j^\varepsilon$ in the sense of distributions in $\Omega \times I$, which comparing with (66) then gives

$$\Xi_{ij} = u_i^\varepsilon u_j^\varepsilon.$$

But

$$b(\boldsymbol{u}_\mu^\varepsilon, \boldsymbol{u}_\mu^\varepsilon, \boldsymbol{w}_j) = -b(\boldsymbol{u}_\mu^\varepsilon, \boldsymbol{w}_j, \boldsymbol{u}_\mu^\varepsilon) \to -b(\boldsymbol{u}_\mu^\varepsilon, \boldsymbol{w}_j, \boldsymbol{u}_\mu^\varepsilon) \text{ weakly in } L^2(I)\ \forall \boldsymbol{w}_j,$$

and similarly for the other trilinear forms.

Then from (51)–(53) for $m=\mu$ we obtain

$$\int_I \{(\partial_t H^\varepsilon(t), z_j(t)) + b_\Theta(\boldsymbol{u}^\varepsilon(t), \Theta^\varepsilon(t), z_j(t))$$

$$+ a_\Theta(\Theta^\varepsilon(t), z_j(t)) + b_p(\boldsymbol{u}^\varepsilon(t), z_j(t)) - (Q(t), z_j(t))\}\,\mathrm{d}t = 0 \quad \forall j,$$

$$\int_I \{(\partial_t \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}_j(t)) + b(\boldsymbol{u}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}_j(t)) + \hat{\mu} a(\boldsymbol{u}^\varepsilon(t), \boldsymbol{v}_j(t))$$

$$+ b_s(\Theta^\varepsilon(t) - \Theta_0, \boldsymbol{v}_j(t)) - (\boldsymbol{F}(t), \boldsymbol{v}_j(t))\}\,\mathrm{d}t + (\chi(t), \boldsymbol{v}_j(t)) = 0 \quad \forall j,$$

$$\int_I \{(\partial_t c^\varepsilon(t), d_j(t)) + b_c(\boldsymbol{u}^\varepsilon(t), c_L^\varepsilon(t), d_j(t))$$

$$+ a_c(c^\varepsilon(t)), d_j(t)) - (\varphi(t), d_j(t))\}\,\mathrm{d}t = 0 \quad \forall j, \tag{67}$$

where we denoted by $\chi(t) = J'(\boldsymbol{u}^\varepsilon(t))$.

But the systems of functions $\{z_j\}$, $\{\boldsymbol{v}_j\}$, $\{w_j\}$ are complete in ${}^1V$ or V or ${}^2V$, respectively, so that from (67) we obtain

$$\int_I \{(\partial_t H^\varepsilon(t), z(t)) + b_\Theta(\boldsymbol{u}^\varepsilon(t), \Theta^\varepsilon(t), z(t)) + a_\Theta(\Theta^\varepsilon(t), z(t))$$

$$+ b_p(\boldsymbol{u}^\varepsilon(t), z(t)) - (Q(t), z(t))\}\,\mathrm{d}t = 0 \quad \forall z \in {}^1V,$$

$$\int_I \{(\partial_t \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t)) + b(\boldsymbol{u}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t)) + \hat{\mu} a(\boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t))$$

$$+ b_s(\Theta^\varepsilon(t) - \Theta_0, \boldsymbol{v}(t)) - (\boldsymbol{F}(t), \boldsymbol{v}(t))\}\,\mathrm{d}t + (\chi(t), \boldsymbol{v}(t)) = 0 \quad \forall \boldsymbol{v} \in V,$$

$$\int_I \{(\partial_t c^\varepsilon(t), d(t)) + b_c(\boldsymbol{u}^\varepsilon(t), c_{\mathrm{L}}^\varepsilon(t), d(t))$$

$$+ a_c(c^\varepsilon(t)), d(t)) - (\varphi(t), d(t))\}\,\mathrm{d}t = 0 \quad \forall d \in {}^2V.$$

Since (47)–(49) and a priori estimates (56) are satisfied, then it is sufficient to prove that

$$\chi(t) = J_\varepsilon'(\boldsymbol{u}^\varepsilon(t)).$$

To prove this, the property of monotonicity will be used. Then the proof is parallel to that of [23].

Let $\boldsymbol{v} \in L^2(I; V)$ such that $\boldsymbol{v} \in L^2(I; V')$, $\boldsymbol{v}(t_0) = \boldsymbol{u}_0$. Let us put

$$X_\mu = (J_\varepsilon'(\boldsymbol{u}_\mu(t)) - J_\varepsilon'(\boldsymbol{v}(t)), \boldsymbol{u}_\mu(t) - \boldsymbol{v}(t)) + \hat{\mu} \int_I a(\boldsymbol{u}_\mu(t) - \boldsymbol{v}(t), \boldsymbol{u}_\mu(t) - \boldsymbol{v}(t))\,\mathrm{d}t$$

$$+ \int_I (\partial_t \boldsymbol{u}_\mu(t) - \partial_t \boldsymbol{v}(t), \boldsymbol{u}_\mu(t) - \boldsymbol{v}(t))\,\mathrm{d}t.$$

Using (48) we have

$$X_\mu = \int_I (\boldsymbol{F}(t), \boldsymbol{u}_\mu(t))\,\mathrm{d}t - (J_\varepsilon'(\boldsymbol{u}_\mu(t)), \boldsymbol{v}(t)) - (J_\varepsilon'(\boldsymbol{v}(t)), \boldsymbol{u}_\mu(t) - \boldsymbol{v}(t))$$
$$-\hat{\mu}\int_I [a(\boldsymbol{u}_\mu(t), \boldsymbol{v}(t)) + a(\boldsymbol{v}(t), \boldsymbol{u}_\mu(t) - \boldsymbol{v}(t))]\,\mathrm{d}t$$
$$-\int_I [(\partial_t \boldsymbol{u}_\mu(t), \boldsymbol{v}(t)) - (\partial_t \boldsymbol{v}(t), \boldsymbol{u}_\mu(t) - \boldsymbol{v}(t))]\,\mathrm{d}t.$$

Hence $X_\mu \to X$, where

$$X = \int_I (\boldsymbol{F}(t), \boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t - [(\chi(t), \boldsymbol{v}(t)) + (J_\varepsilon'(\boldsymbol{v}(t)), \boldsymbol{u}^\varepsilon(t) - \boldsymbol{v}(t))]$$
$$-\hat{\mu}\int_I [a(\boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t)) + a(\boldsymbol{v}(t), \boldsymbol{u}^\varepsilon(t) - \boldsymbol{v}(t))]\,\mathrm{d}t$$
$$-\int_I (\partial_t \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t)) - (\partial_t \boldsymbol{v}(t), \boldsymbol{u}^\varepsilon(t) - \boldsymbol{v}(t))]\,\mathrm{d}t.$$

Since $X_\mu(t) \geqslant 0$ for all μ, then $X(t) \geqslant 0$.

Let us put $\boldsymbol{v} = \boldsymbol{u}^\varepsilon - \lambda \boldsymbol{w}$, $\lambda \geqslant 0$, where $\boldsymbol{w} \in L^2(I; V)$, $\partial_t \boldsymbol{w} \in L^2(I; V')$, $\boldsymbol{w}(t_0) = 0$. Substituting for $\boldsymbol{v}$ dividing by λ we obtain

$$(\chi(t) - J_\varepsilon'(\boldsymbol{u}^\varepsilon(t) - \lambda \boldsymbol{w}(t)), \boldsymbol{w}(t)) + \lambda \int_{\mathrm{I}} \{\hat{\mu} a(\boldsymbol{w}(t), \boldsymbol{w}(t)) + (\partial_t \boldsymbol{w}(t), \boldsymbol{w}(t))\}\,\mathrm{d}t \geqslant 0.$$

Hence, in the limit $\lambda \to 0$ we obtain

$$(\chi(t) - J_\varepsilon'(\boldsymbol{u}^\varepsilon(t)), \boldsymbol{w}(t)) \geqslant 0 \quad \forall \boldsymbol{w},$$

from which the assertion $\chi(t) = J_\varepsilon'(\boldsymbol{u}^\varepsilon(t))$ follows.

Therefore, we proved the existence of H^ε, $\boldsymbol{u}^\varepsilon$, c^ε satisfying (47)–(49) and the conditions

$$\begin{array}{lll}
H^\varepsilon(t) & \text{is bounded in} & L^2(I; {}^1V) \cap H^1(I; H^{-1}(\Omega)), \\
\partial_t H^\varepsilon(t) & \text{is bounded in} & L^2(I; H^{-1}(\Omega)), \\
\boldsymbol{u}^\varepsilon(t) & \text{is bounded in} & L^2(I; V) \cap L^\infty(I; H), \\
\partial_t \boldsymbol{u}^\varepsilon(t) & \text{is bounded in} & L^2(I; V'), \\
c^\varepsilon(t) & \text{is bounded in} & L^2(I; {}^2V) \cap H^1(I; (H^1(\Omega))'), \\
\partial_t c^\varepsilon(t) & \text{is bounded in} & L^2(I; (H^1(\Omega))').
\end{array}$$

To prove that $H^\varepsilon(\boldsymbol{x}, t_0) = H_0(\boldsymbol{x})$, $\boldsymbol{u}^\varepsilon(\boldsymbol{x}, t_0) = \boldsymbol{u}_0(\boldsymbol{x})$, $c^\varepsilon(\boldsymbol{x}, t_0) = c_0(\boldsymbol{x})$ the Arzelà-Ascoli's theorem[7] will be used. Due to this theorem, the sequences $\{H_\mu^\varepsilon\}$, $\{\boldsymbol{u}_\mu^\varepsilon\}$, $\{c_\mu^\varepsilon\}$, $\mu \in \mathcal{N}$ converge toward $H^\varepsilon \in$

[7] Let K be a subset of $C^0(I; X)$, X is a Banach space. Then K is said to be equi-continuous if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|u(t) - u(s)\|_X < \varepsilon$ holds for all $u \in K$ and all t, $s \in I$ for which $|t - s| < \delta$.

Arzelà-Ascoli theorem: A subset K of $C^0(I; K)$ is relatively compact if and only if (i) K is equi-continuous, (ii) the set $K(t)$, $K(t) = \{u(t) \mid u \in K\}$, is relatively compact in Banach space X for every $t \in I$, i.e. every sequence in $K(t)$ contains a subsequence convergent in X.

$C^0(I;H^{-1}(\Omega))$ and/or toward $\boldsymbol{u}^\varepsilon \in C^0(I;V')$, and/or toward $c^\varepsilon \in C^0(I;(H^1(\Omega))')$, respectively, and therefore $\{H^\varepsilon_\mu(.,t_0)\}$, $\{\boldsymbol{u}^\varepsilon_\mu(.,t_0)\}$, $\{c^\varepsilon_\mu(.,t_0)\}$, $\mu \in \mathscr{N}$, converge toward $H^\varepsilon(.,t_0)$ in $H^{-1}(\Omega)$, or toward $\boldsymbol{u}^\varepsilon(.,t_0) \in V'$, or toward $c^\varepsilon(.,t_0) \in (H^1(\Omega))'$, respectively, and at the same time $\{H^\varepsilon_\mu(.,t_0)\}$, $\mu \in \mathscr{N}$, converges to $H_0(\boldsymbol{x})$ in $L^2(\Omega)$ by definition. Similarly, for $\{\boldsymbol{u}^\varepsilon_\mu(.,t_0)\}_{\mu\in\mathscr{N}} \to \boldsymbol{u}_0(\boldsymbol{x})$ and $\{c^\varepsilon_\mu(.,t_0)\}_{\mu\in\mathscr{N}} \to c_0(\boldsymbol{x})$ in $H(\Omega)$ or $L^2(\Omega)$, respectively, by the definition. Since $H^\varepsilon \in C^0(I;L^2(\Omega))$, the initial condition holds in $L^2(\Omega)$ and therefore a.e. in Ω. By a similar way we finish the proof for $\boldsymbol{u}_0(\boldsymbol{x})$ and $c_0(\boldsymbol{x})$.

5.4. Limitation process $\varepsilon \to 0$

Now we shall prove that estimates of regularized enthalpy H^ε and its derivative $\partial_t H^\varepsilon$ as well as regularized temperature T^ε do not depend on ε. The proof of this is parallel to that of Lemma 3.4 of [1] (see also [2]).

First, we prove the auxiliary estimate used later. Let assumptions (A2)(iii), (A4)(iii) be satisfied. Let α satisfy (A2), (A4). Let $\varphi = \alpha'(c^\varepsilon + \varepsilon)$ be a test function (it is possible as $c^\varepsilon \geqslant 0$ in $\Omega \times I$ and α' is uniformly Lipschitzian on $\langle \varepsilon, \infty))$. Then (49c) yields

$$\int_\Omega \alpha(c^\varepsilon(.,t_1)+\varepsilon)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \alpha(c_0+\varepsilon)\,\mathrm{d}\boldsymbol{x} + \int_I\int_\Omega \alpha''(c^\varepsilon+\varepsilon)\,\|\mathrm{grad}\ c^\varepsilon\|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$-\int_I\int_\Omega c^\varepsilon_{\mathrm{L}}(c^\varepsilon, T^\varepsilon(c^\varepsilon,H^\varepsilon))\alpha''(c^\varepsilon+\varepsilon)\boldsymbol{u}^\varepsilon\,\mathrm{grad}\ c^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0.$$

Hence, due to the Young inequality and the property of α as well as Lemma 3(iv), we then obtain

$$\begin{aligned}
\int_I\int_\Omega \alpha''(c^\varepsilon+\varepsilon)\,\|\mathrm{grad}\ c^\varepsilon\|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &\leqslant \int_\Omega \alpha(c_0+\varepsilon)\,\mathrm{d}\boldsymbol{x} - \inf_{\mathbb{R}_+}(\alpha)\,\mathrm{meas}(\Omega)\\
&\quad + \int_I\int_\Omega |c^\varepsilon_{\mathrm{L}}(c^\varepsilon, T^\varepsilon(c^\varepsilon,H^\varepsilon))|\alpha''(c^\varepsilon+\varepsilon)\,\|\boldsymbol{u}^\varepsilon\|\,\|\mathrm{grad}\ c^\varepsilon\|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\leqslant c + \frac{1}{2}\int_I\int_\Omega \alpha''(c^\varepsilon+\varepsilon)\,\|\mathrm{grad}\ c^\varepsilon\|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad + \frac{1}{2}\,\|\boldsymbol{u}^\varepsilon\|_{L^\infty(\Omega)}\int_I\int_\Omega |c^\varepsilon_{\mathrm{L}}(c^\varepsilon, T^\varepsilon(c^\varepsilon,H^\varepsilon))|^2\alpha''(c^\varepsilon+\varepsilon)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leqslant c.
\end{aligned} \tag{68}$$

Now, let $t \in I$ be an arbitrary. Then (47) for $z = \hat{\chi}(t)H^\varepsilon$, where $\hat{\chi}(t)$ is a characteristic function of the interval I, yields

$$\begin{aligned}
0 &= \frac{1}{2}\int_\Omega (H^\varepsilon(.,t))^2\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (H_0)^2\,\mathrm{d}\boldsymbol{x}\\
&\quad + \int_I\int_\Omega \boldsymbol{u}^\varepsilon \mathrm{grad}\ T^\varepsilon(c^\varepsilon,H^\varepsilon)(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-\frac{1}{2}}(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{1/2}H^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\quad + \int_I\int_\Omega \mathrm{grad}\ T^\varepsilon(c^\varepsilon,H^\varepsilon)[\mathrm{grad}\ T^\varepsilon(c^\varepsilon,H^\varepsilon)\\
&\quad - \partial_1 T^\varepsilon(c^\varepsilon,H^\varepsilon)\mathrm{grad}\ c^\varepsilon](\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.
\end{aligned}$$

Hence, after some modifications

$$\frac{1}{2}\int_\Omega (H^\varepsilon(.,t))^2\,\mathrm{d}\boldsymbol{x} + \int_I\int_\Omega \operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)\operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$= \frac{1}{2}\int_\Omega (H_0)^2\,\mathrm{d}\boldsymbol{x} - \int_I\int_\Omega \operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\partial_1 T^\varepsilon(c^\varepsilon,H^\varepsilon)\operatorname{grad} c^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \int_I\int_\Omega \operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1/2}\boldsymbol{u}^\varepsilon(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{1/2}H^\varepsilon\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

and after applying twice the Young inequality, we find

$$\frac{1}{2}\int_\Omega (H^\varepsilon(.,t))^2\,\mathrm{d}\boldsymbol{x} + \int_I\int_\Omega \|\operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)\|^2\,(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\leqslant \frac{1}{2}\int_\Omega (H_0)^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{4}\int_I\int_\Omega \|\operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)\|^2\,(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \int_I\int_\Omega |\partial_1 T^\varepsilon(c^\varepsilon,H^\varepsilon)|^2(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\,\|\operatorname{grad} c^\varepsilon\|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \frac{1}{4}\int_I\int_\Omega \|\operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)\|^2\,(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ \|\boldsymbol{u}^\varepsilon\|_{L^\infty(\Omega\times I)}\|\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon)\|_{L^\infty(\Omega\times I)}\int_I\int_\Omega (H^\varepsilon)^2\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Hence

$$\int_\Omega (H^\varepsilon(.,t))^2\,\mathrm{d}\boldsymbol{x} + \int_I\int_\Omega \|\operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)\|^2\,(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$\leqslant \int_\Omega (H_0^\varepsilon))^2\,\mathrm{d}\boldsymbol{x} + 2\int_I\int_\Omega |\partial_1 T^\varepsilon(c^\varepsilon,H^\varepsilon)|^2(\partial_2 T^\varepsilon(c^\varepsilon,H^\varepsilon))^{-1}\,\|\operatorname{grad} c^\varepsilon\|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$+ c\int_I\int_\Omega (H^\varepsilon)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Hence, using Lemma 3(v), (68) and the Gronwall lemma, we obtain

$$\|H^\varepsilon\|_{L^\infty(I;L^2(\Omega))} \leqslant c, \quad c = \text{const. independent of } \varepsilon. \tag{69}$$

Since $\partial_2 T^\varepsilon$ is bounded from above, then

$$\|T^\varepsilon(c^\varepsilon,H^\varepsilon)\|_{L^2(I;H_0^1(\Omega))} \leqslant c, \quad c = \text{const. independent of } \varepsilon. \tag{70}$$

From (47) it follows

$$\int_I (\partial_t H^\varepsilon, z)\,\mathrm{d}t = -\int_I (\boldsymbol{u}^\varepsilon \operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon), z)\,\mathrm{d}t - \int_I a_\Theta(T^\varepsilon(c^\varepsilon,H^\varepsilon), z)\,\mathrm{d}t$$
$$\leqslant \|\boldsymbol{u}^\varepsilon\|_{L^\infty(\Omega\times I)}\int_I \|\operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)\|_{0,1}\|z\|_{0,1}\,\mathrm{d}t$$
$$+ \int_I \|\operatorname{grad} T^\varepsilon(c^\varepsilon,H^\varepsilon)\|_{0,1}\|\operatorname{grad} z\|_{0,1}\,\mathrm{d}t \quad \forall z \in L^2(I;H_0^1(\Omega)).$$

Since

$$\|z\|^2_{L^2(I;H^{-1}(\Omega))} = \int_I \left(\sup_{w\in H_0^1(\Omega),\|w\|_{H_0^1(\Omega)}\leqslant 1} (z,w) \right)^2 \mathrm{d}t$$

and using (70), then

$$\|\partial_t H^\varepsilon\|_{L^2(I;H^{-1}(\Omega))} \leqslant c, \quad c = \text{const. independent of } \varepsilon. \tag{71}$$

The existence of a weak solution of the problem $(\mathscr{P}_{\text{centh}})_v$ will be established as the limit of a subsequence of weak solutions of the regularized problems. In the previous steps we proved that

$$\begin{aligned}
&H^\varepsilon && \text{remains in a bounded set of} && L^\infty(I;L^2(\Omega)),\\
&\partial_t H^\varepsilon && \text{remains in a bounded set of} && L^\infty(I;H^{-1}(\Omega)),\\
&T^\varepsilon && \text{remains in a bounded set of} && L^2(I;H_0^1(\Omega)),\\
&\boldsymbol{u}^\varepsilon && \text{remains in a bounded set of} && L^2(I;V)\cap L^\infty(I;H),\\
&\partial_t \boldsymbol{u}^\varepsilon && \text{remains in a bounded set of} && L^\infty(I;V'),\\
&c^\varepsilon && \text{remains in a bounded set of} && L^\infty(I;H^1(\Omega)),\\
&\partial_t c^\varepsilon && \text{remains in a bounded set of} && L^\infty(I;(H^1(\Omega))')
\end{aligned} \tag{72}$$

for any weak regularized solution $\{H^\varepsilon, \Theta^\varepsilon, \boldsymbol{u}^\varepsilon, c^\varepsilon\}$.

In this part of the proof we shall prove that

$$\begin{aligned}
&H^\varepsilon \to H \quad \text{in} \quad C^0(I;H^{-1}(\Omega)),\\
&T^\varepsilon \to T \quad \text{strongly in} \quad C^0(I;L^2(\Omega)),\\
&\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad *\text{-weakly in} \quad L^\infty(I;H) \quad \text{and weakly in} \quad L^2(I;V),\\
&\partial_t \boldsymbol{u}^\varepsilon \to \partial_t \boldsymbol{u} \quad \text{weakly in} \quad L^2(I;V'),\\
&c^\varepsilon \to c \quad \text{strongly in} \quad C^0(I;L^2(\Omega)).
\end{aligned}$$

Due to (72) the set $\{H^\varepsilon(t), 0<\varepsilon\leqslant 1\}$ is relatively compact in $H^{-1}(\Omega)$ $\forall t\in\bar{I}$, and then the family $\{H^\varepsilon \mid 0<\varepsilon\leqslant 1\}$ is equi-continuous in $H^{-1}(\Omega)$. Then due to the Arzela-Ascoli theorem there exists a subsequence $\{H^{\varepsilon_j}\}$, $j\in\mathscr{N}$, converging in $C^0(I;H^{-1}(\Omega))$. Moreover, due to (72) there exists a subsequence $\{\boldsymbol{u}^{\varepsilon_j}\}$, $j\in\mathscr{N}$, such that

$$\begin{aligned}
&\boldsymbol{u}^{\varepsilon_j}(t) \to \boldsymbol{u}(t) \quad \text{in} \quad L^\infty(I;H) \quad \text{weakly star and weakly in} \quad L^2(I;V),\\
&\partial_t \boldsymbol{u}^{\varepsilon_j}(t) \to \partial_t \boldsymbol{u}(t) \quad \text{weakly in} \quad L^2(I;V').
\end{aligned} \tag{73}$$

For $\boldsymbol{v}\in W$ let us put (the index j will be omitted)

$$\begin{aligned}
Y_\varepsilon = \int_I \{&(\partial_t \mathrm{v}(t), \boldsymbol{v}(t)-\boldsymbol{u}^\varepsilon(t)) + \hat{\mu} a(\boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t)-\boldsymbol{u}^\varepsilon(t))\\
&+ b(\boldsymbol{u}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t)-\boldsymbol{u}^\varepsilon(t)) + b_s(\Theta^\varepsilon(t)-\Theta_0, \boldsymbol{v}(t)-\boldsymbol{u}^\varepsilon(t))\\
&- (\boldsymbol{F}(t), v(t)-\boldsymbol{u}^\varepsilon(t))\}\,\mathrm{d}t + J_\varepsilon(\boldsymbol{v}(t)) - J_\varepsilon(\boldsymbol{u}^\varepsilon(t)).
\end{aligned}$$

By virtue of (48),

$$Y_\varepsilon = \int_I (\partial_t \boldsymbol{v}(t) - \partial_t \boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t) - \boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t + J_\varepsilon(\boldsymbol{v}(t))$$
$$-J_\varepsilon(\boldsymbol{u}^\varepsilon(t)) - (J'_\varepsilon(\boldsymbol{u}^\varepsilon(t)), \boldsymbol{v}(t) - \boldsymbol{u}^\varepsilon(t)).$$

Due to the initial conditions the first term is equivalent to $\frac{1}{2}|\boldsymbol{v}(t_1) - \boldsymbol{u}^\varepsilon(t_1)|^2$ and since the functional $\boldsymbol{v} \to j_\varepsilon(\boldsymbol{v})$ is convex, the second term $\geqslant 0$, and thus $Y_\varepsilon \geqslant 0$. Hence

$$\int_I \{(\partial_t \boldsymbol{v}(t), \boldsymbol{v}(t) - \boldsymbol{u}^\varepsilon(t)) + \hat{\mu} a(\boldsymbol{u}^\varepsilon(t), v(t)) - b(\boldsymbol{u}^\varepsilon(t), \boldsymbol{v}(t), \boldsymbol{u}^\varepsilon(t))$$
$$+ b_s(\Theta^\varepsilon(t) - \Theta_0, \boldsymbol{v}(t) - \boldsymbol{u}^\varepsilon(t)) - (\boldsymbol{F}(t), \boldsymbol{v}(t) - \boldsymbol{u}^\varepsilon(t))]\,\mathrm{d}t + J_\varepsilon(\boldsymbol{v}(t))$$
$$\geqslant \hat{\mu} \int_I a(\boldsymbol{u}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t + J_\varepsilon(\boldsymbol{u}^\varepsilon(t)).$$

Using this result and (73)

$$\int_I \{(\partial_t \boldsymbol{u}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)) + \hat{\mu} a(\boldsymbol{u}(t), \boldsymbol{v}(t)) - b(\boldsymbol{u}(t), \boldsymbol{v}(t), \boldsymbol{u}(t))$$
$$+ b_s(\Theta(t) - \Theta_0, \boldsymbol{v}(t) - \boldsymbol{u}(t)) - (\boldsymbol{F}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t))\}\,\mathrm{d}t + J(\boldsymbol{v}(t))$$
$$\geqslant \liminf \left(\int_I [\hat{\mu} a(\boldsymbol{u}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t)) + \hat{g} j_\varepsilon(\boldsymbol{u}^\varepsilon(t))] \right) \mathrm{d}t$$
$$\geqslant \int_I \hat{\mu} a(\boldsymbol{u}(t), \boldsymbol{u}(t))\,\mathrm{d}t + J(\boldsymbol{u}(t)), \tag{74}$$

as $\liminf \int_I [\hat{\mu} a(\boldsymbol{u}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t))]\,\mathrm{d}t \geqslant \int_I [\hat{\mu} a(\boldsymbol{u}(t), \boldsymbol{u}(t))]\,\mathrm{d}t$ (due to the fact that the function $\boldsymbol{u} \to \int_I \hat{\mu} a(\boldsymbol{u}, \boldsymbol{u})\mathrm{d}t$ is lower semi-continuous on $L^2(I; V)$ with the weak topology) and since

$$\int_I j(\boldsymbol{u}(t))\,\mathrm{d}t \leqslant \left(\int_{\Omega \times I} D_{\mathrm{II}}(\boldsymbol{u}(t))^{\frac{1}{2}(1+\varepsilon)}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right)^{1/(1+\varepsilon)} \left(\int_{\Omega \times I} \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right)^{\varepsilon/(1+\varepsilon)},$$

hence

$$\int_I j_\varepsilon(\boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t \geqslant c(\varepsilon) \left(\int_I j(\boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t \right)^{(1+\varepsilon)}, \qquad c(\varepsilon) = |\mathrm{meas}(\Omega \times I)|^{-\varepsilon},$$
$$\liminf \int_I j_\varepsilon(\boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t \geqslant \liminf \int_I j(\boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t. \tag{75}$$

Since the function $\boldsymbol{v} \to \int_I j(\boldsymbol{v}(t))\,\mathrm{d}t$ is convex and continuous on $L^2(I; V)$, then it is lower semi-continuous in the weak topology of the space $L^2(I; V)$ and thus

$$\liminf \int_I j(\boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t \geqslant \int_I j(\boldsymbol{u}(t))\,\mathrm{d}t,$$

which together with (75) proves

$$\liminf \int_I j_\varepsilon(\boldsymbol{u}^\varepsilon(t))\,\mathrm{d}t \geqslant \int_I j(\boldsymbol{u}(t))\,\mathrm{d}t.$$

Then (73)–(75) yield that $\boldsymbol{u}$ satisfies (43).

Furthermore, due to (72f, g) a subsequence $\{c^{\varepsilon_j}\}$, $j \in \mathcal{N}$ exists, which strongly converges to $c(t)$ in $C^0(I; L^2(\Omega))$.

Now we shall prove the existence of a subsequence of $\{T^{\varepsilon_j}(c^{\varepsilon_j}, H^{\varepsilon_j})\}$, $j \in \mathcal{N}$, which strongly converges in $L^2(I; L^2(\Omega))$ towards $T(c,H)$. The monotonicity arguments (see [2, 3]) will be used. Previously we proved that $H^{\varepsilon_j} \to H$ in $C^0(I; H^{-1}(\Omega))$, $c^{\varepsilon_j} \to c$ strongly in $L^2(I; L^2(\Omega))$, $j \to \infty$. Let $\{H^j, c^j\}$, $j \in \mathcal{N}$, be the corresponding sequence and $\{T^j(c^j, H^j)\}$, $j \in \mathcal{N}$, the associated sequence. Further, for simplicity we shall write H^j, c^j instead of H^{ε_j} and c^{ε_j}, respectively. Then, due to Lemmata 4.1 of [1], there exist subsequences, still indexed by j, and null measure $E \subset I$ such that, for any $t \in I \backslash E$

$$\lim_{j\to\infty} \int_\Omega T^j(c^j(t), H^j(t))(H^j(t) - H(t))\,\mathrm{d}\Omega = 0, \tag{76}$$

$$\lim_{j\to\infty} \int_\Omega |c^j(t) - c(t)|^2\,\mathrm{d}\Omega = 0 \quad \forall t \in I \backslash E.$$

Due to assumptions (A2)(i), (ii) and (72), then a subsequence $\{T^k(c^k(.,t), H^k(.,t))\}$, $k \in \mathcal{N}$ exists, which is bounded in $L^2(\Omega)$. Hence, for every $t \in I \backslash E$ a subsequence $\{T^l(c^l(.,t), H^l(.,t))\}$, $l \in \mathcal{N}$ exists, dependent on t such that (see Lemmata 4.2. of [1])

$$T^l(c^l(.,t), H^l(.,t)) \to T(c(.,t), H(.,t)) \quad \text{weakly in} \quad L^2(\Omega). \tag{77}$$

It remains to prove that $T^j(c^j, H^j) \to T(c,H)$ strongly in $L^2(I; L^2(\Omega))$.

Let $\{T^j(c^j, H^j)\}$, $j \in \mathcal{N}$, be the subsequence resulting from the associated sequence defined above. Then, firstly we prove that

$$f^j \equiv T^j(c^j, H^j) - T(c,H) \to 0 \quad \text{weakly in} \quad L^2(\Omega), \quad \forall t \in I \backslash E, \tag{78}$$

where E is a null measure set defined above.

We prove (78) by contradiction. Let $\{f^j\}$, $j \in \mathcal{N}$, be a subsequence, $f \neq 0$, $f \in L^2(\Omega)$ function. Then, there can not be $t \in I \backslash E$, a subsequence $\{f^j\}$ and a function $f \neq 0$, $f \in L^2(\Omega)$ does not exist, such that $\{f^k\}$, $k \in \mathcal{N}$, weakly converges towards f in $L^2(\Omega)$. Thus no subsequence $\{f^k\} \to 0$ weakly, $k \in \mathcal{N}$. But it is in contradiction with (77). Hence, we deduce the existence of a subsequence $\{f^j\}$, $j \in \mathcal{N}$, weakly converging towards 0.

Let $\{w_k\}$, $k \in \mathcal{N}$, be a basis of $L^2(\Omega)$. Since for almost every $t \in I \backslash E$, the functions $f^j \in H^1(\Omega)$, then due to Fridrichs' lemma [8] for any $\varepsilon > 0$

$$\| f^j(.,t) \|^2_{L^2(\Omega)} \leqslant \varepsilon \, \| f^j(.,t) \|^2_{H^1_0(\Omega)} + \sum_{k=0}^{n} \left(\int_\Omega f^j(x,t) w_k(x)\,\mathrm{d}x \right)^2, \quad n = n(\varepsilon).$$

Hence, by integrating over $t \in I$ and putting $\alpha^{k,j}(t) = (\int_\Omega f^j(x,t)\; w_k(x)\,\mathrm{d}x)^2$ we first, due to estimate (68) and the weak convergence of $\{f^j\}$, $j \in \mathcal{N}$, find that $\{\alpha^{k,j}(t)\}$, $j \in \mathcal{N}$ converges towards

[8] *Fridrich's lemma:* Let Ω be a domain with a Lipschitz boundary Γ. Then non-negative constants c_1, c_2 exist, which are dependent on the considered domain but independent of the functions from $M =$ (the linear set of functions $u(x)$ which are continuous with their partial derivatives of the 1st order in $\overline{\Omega}$ (i.e. the set $C^1(\overline{\Omega})$)}, such that $\int_\Omega (u(x))^2\,\mathrm{d}x \leqslant c_1 \sum_{i=1}^N \int_\Omega (\partial_i u)^2\,\mathrm{d}x + c_2 \int_\Gamma (u(s))^2\,\mathrm{d}s$ holds for every function $u \in M$.

0 in $L^1(I)$. Secondly, then a constant c, independent of ε, exists such that

$$\lim_{j\to\infty} \|f^j\|_{L^2(\Omega\times I)} \equiv \lim_{j\to\infty} \|T^j(c^j,H^j) - T(c,H)\|_{L^2(\Omega\times I)} \leqslant c\varepsilon.$$

Since the last estimate is valid for any $\varepsilon > 0$, then

$$T^j(c^j,H^j) \to T(c,H) \quad \text{strongly in} \quad L^2(I;L^2(\Omega)).$$

Summarizing all of these results we find that if $(H^\varepsilon, \boldsymbol{u}^\varepsilon, c^\varepsilon)$, $\varepsilon \in (0,1\rangle$ is a weak regularized solution, then there is a sequence indexed by j and a tetrad $\{H,\Theta,\boldsymbol{u},c\}$ satisfying the regularity conditions in Definition 1 such that

$$\begin{array}{lll}
\partial_t H^j \to \partial_t H & \text{weakly in} & L^2(I;H^{-1}(\Omega)),\\
\partial_t c^j \to \partial_t c & \text{weakly in} & L^2(I;(H^1(\Omega))'),\\
\operatorname{grad} c^j \to \operatorname{grad} c & \text{weakly in} & L^2(I;L^2(\Omega)),\\
\operatorname{grad} T^j(c^j,H^j) \to \operatorname{grad} \Theta & \text{weakly in} & L^2(I;L^2(\Omega)),\\
c_{\mathrm{L}}(c^j, T(c^j,H^j)) \to c_{\mathrm{L}}(c,T) & \text{strongly in} & L^2(I;L^2(\Omega)),
\end{array}$$

where $\Theta = T(c,H)$.

Due to [7] for $\hat{g} = 0$ it follows that the Bingham rheology leads to the Newtonian liquid, and then

$$\begin{array}{l}
\boldsymbol{u} \to \boldsymbol{u}_{\mathrm{L}} \quad \text{weakly in} \quad L^2(I;V),\\
\partial_t \boldsymbol{u} \to \partial_t \boldsymbol{u}_{\mathrm{L}} \quad \text{weakly in} \quad L^2(I;V'),
\end{array} \tag{79}$$

where by $\boldsymbol{u}_{\mathrm{L}}$ we denoted the solution corresponding to the liquid state. The proof of (79) is parallel of that of Theorem 5.1. of [7, pp. 306–307], where in addition $b_s(\Theta - \Theta_0, \boldsymbol{v}) \in L^2(I;[L^2(\Omega)]^2)$ was assumed to be a part of body forces.

Now we prove this assertion in more details. We proved above that for $\hat{g} > 0$ a weak solution of a coupled two-Stefan-like problem exists in a Bingham visco-plastic rheology. It was shown, that in the mushy zones both phases, solid (visco-plastic) and liquid (strongly visco-plasic with low $\hat{g}$) are microscopically parallelly present. Therefore, we must show that for the threshold of plasticity g tends to zero, i.e. if $\hat{g} \to 0$, the Bingham visco-plastic rheology leads to the Newtonian viscous fluid. This means (79) is proved if $\hat{g} \to 0$.

We proved above that

$$\|\boldsymbol{u}\|_{L^2(I;V)} + \|\partial_t \boldsymbol{u}\|_{L^2(I;V')} \leqslant c \quad \text{for all } \hat{g} > 0,$$

where $\hat{g}$ is bounded. Now, let us assume that $\hat{g} \to 0$. Then we can take out a sequence, we denote it again by $\{\boldsymbol{u}\}$, such that

$$\begin{array}{lll}
\boldsymbol{u} \to \boldsymbol{w} & \text{in } L^2(I;V) & \text{weakly},\\
\partial_t \boldsymbol{u} \to \partial_t \boldsymbol{w} & \text{in } L^2(I;V') & \text{weakly},\\
u_i \to w_i & \text{in } L^2(\Omega \times I) & \text{strongly in } L^2(I;L^2(\Omega)) \text{ and a.e. in } \Omega \times I,\\
u_i u_j \to w_i w_j & \text{in} L^2(I;L^2(\Omega)), &
\end{array} \tag{80}$$

where u_i, w_i are components of $\boldsymbol{u}$ and/or $\boldsymbol{w}$, where $\boldsymbol{w}$ is a solution of the Navier–Stokes problem: $\boldsymbol{u}_L \in L^2(I;V)$, $\partial_t \boldsymbol{u}_L \in L^2(I;V')$

$$(\partial_t \boldsymbol{u}_L(t), \boldsymbol{v}) + \hat{\mu} a(\boldsymbol{u}_L(t), \boldsymbol{v}) + b(\boldsymbol{u}_L(t), \boldsymbol{u}_L(t), \boldsymbol{v}) = (\boldsymbol{F}(t), v) \quad \forall \boldsymbol{v} \in V,$$
$$\boldsymbol{u}_L(\boldsymbol{x}, t_0) = \boldsymbol{u}_0(\boldsymbol{x}), \tag{81}$$

where the coupled term was included into the body forces as above. Then, similarly as above

$$b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = -b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}) \to -b(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{w}) \quad \text{in } L^2(I) \text{ weakly}, \quad \forall \boldsymbol{v} \in V. \tag{82}$$

Let us put $\boldsymbol{v} = \boldsymbol{v}(t)$, $\boldsymbol{v} \in L^2(I;V)$, then

$$\int_I [(\partial_t \boldsymbol{u}(t), v(t) - \boldsymbol{u}(t)) + \hat{\mu} a(\boldsymbol{u}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t)) + b(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}(t))$$
$$- (\boldsymbol{F}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t))] \, \mathrm{d}t + J(\boldsymbol{v}(t)) - J(\boldsymbol{u}(t)) \geqslant 0,$$

then

$$\int_I [(\partial_t \boldsymbol{u}(t), v(t)) + \hat{\mu} a(\boldsymbol{u}(t), \boldsymbol{v}(t)) - b(\boldsymbol{u}(t), \boldsymbol{v}(t), \boldsymbol{u}(t))$$
$$- (\boldsymbol{F}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t))] \, \mathrm{d}t + J(\boldsymbol{v}(t)) - J(\boldsymbol{u}(t))$$
$$\geqslant \int_I [(\partial_t \boldsymbol{u}(t), \boldsymbol{u}(t)) + \hat{\mu} a(\boldsymbol{u}(t), \boldsymbol{u}(t))] \, \mathrm{d}t \geqslant \tfrac{1}{2} |\boldsymbol{u}(t_1)|^2 - \tfrac{1}{2} |\boldsymbol{u}_0|^2 + \hat{\mu} \int_I a(\boldsymbol{u}(t), \boldsymbol{u}(t))] \, \mathrm{d}t.$$

Hence, (80), (82) and since $\int_I \hat{g} j(\boldsymbol{u}(t)) \, \mathrm{d}t \to 0$, we obtain

$$\int_I [(\partial_t \boldsymbol{w}, \boldsymbol{v}) + \hat{\mu} a(\boldsymbol{w}, \boldsymbol{v}) - b(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{w}) - (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{w})] \, \mathrm{d}t$$
$$\geqslant \liminf \frac{1}{2} |\boldsymbol{u}(t_1)|^2 - \frac{1}{2} |\boldsymbol{u}_0|^2 + \hat{\mu} \liminf \int_I a(\boldsymbol{u}(t), \boldsymbol{u}(t))] \, \mathrm{d}t$$
$$\geqslant \frac{1}{2} |\boldsymbol{w}(t_1)|^2 - \frac{1}{2} |\boldsymbol{u}_0|^2 + \hat{\mu} \int_I a(\boldsymbol{w}(t), \boldsymbol{w}(t))] \, \mathrm{d}t$$
$$= \int_I [(\partial_t \boldsymbol{w}, \boldsymbol{w}) + \hat{\mu} a(\boldsymbol{w}(t), \boldsymbol{w}(t))] \, \mathrm{d}t.$$

Hence,

$$\int_I [(\partial_t \boldsymbol{w}(t), \boldsymbol{v}(t) - \boldsymbol{w}(t)) + \hat{\mu} a(\boldsymbol{w}(t), \boldsymbol{v}(t) - \boldsymbol{w}(t)) - b(\boldsymbol{w}(t), \boldsymbol{v}(t), \boldsymbol{w}(t))$$
$$- (\boldsymbol{F}(t), \boldsymbol{v}(t) - \boldsymbol{w}(t))] \, \mathrm{d}t \geqslant 0 \quad \forall \boldsymbol{v} \in L^2(I;V)$$

and therefore, for almost everywhere on $\bar{I}$

$$(\partial_t \boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t)) + \hat{\mu} a(\boldsymbol{w}(t), \boldsymbol{v} - \boldsymbol{w}(t)) - b(\boldsymbol{w}(t), \boldsymbol{v}, \boldsymbol{w}(t)) - (\boldsymbol{F}(t), \boldsymbol{v} - \boldsymbol{w}(t)) \geqslant 0 \quad \forall \boldsymbol{v} \in V.$$

Hence, putting $\boldsymbol{v}=\boldsymbol{w}(t)\pm\boldsymbol{\omega}$, $\boldsymbol{\omega}\in V$, it follows

$$(\partial_t \boldsymbol{w}(t),\boldsymbol{\omega})+\hat{\mu}a(\boldsymbol{w}(t),\boldsymbol{\omega})-b(\boldsymbol{w}(t),\boldsymbol{\omega},\boldsymbol{w}(t))-(\boldsymbol{F}(t),\boldsymbol{\omega}))\geqslant 0 \quad \forall\boldsymbol{\omega}\in V. \tag{83}$$

As it is known (see, e.g., [36]) that the Navier–Stokes problem (81) has a unique solution, then from (83) it follows that

$$\boldsymbol{w}(t)=\boldsymbol{u}_L(t),$$

which proves (79). Moreover, due to (67) it follows that

$$\partial_t \boldsymbol{u}(t)\to\partial_t \boldsymbol{u}_L(t) \quad \text{weakly in } L^2(I;V) \quad \text{and} \quad *\text{-weakly in } L^2(I;H). \tag{84}$$

At the end we shall prove that functions $m=\{m_{ij}\}$ exist, the so-called multipliers, satisfying assumptions (A3), which continuously depend on $\hat{g}$ in a suitable topology. For this case from (43) we have

$$\begin{aligned}&\int_I \{(\partial_t \boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))+\hat{\mu}a(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))+b(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))\\&\qquad +b_s(\Theta(t)-\Theta_0,\boldsymbol{v}-\boldsymbol{u}(t))-(\boldsymbol{F}(t),\boldsymbol{v}-\boldsymbol{u}(t))\}\,\mathrm{d}t+J(\boldsymbol{v})-J(\boldsymbol{u}(t))\\&\quad =J(\boldsymbol{v})-J(\boldsymbol{u}(t))-2\hat{g}\int_I[(m_{ij},D_{ij}(\boldsymbol{v})-D_{ij}(\boldsymbol{u}))]\,\mathrm{d}t\equiv X\geqslant 0 \quad \forall\boldsymbol{v}\in W,\end{aligned} \tag{85}$$

due to the assumptions (A3)(ii),(iii), $j(\boldsymbol{u})=2(m_{ij},D_{ij}(\boldsymbol{u}))$, we find

$$2(m_{ij},D_{ij}(\boldsymbol{v}))\leqslant 2\int_\Omega (D_{ij}(\boldsymbol{v})D_{ij}(\boldsymbol{v}))^{1/2}\,\mathrm{d}\boldsymbol{x}=j(\boldsymbol{v})$$

and therefore $X\geqslant 0$.

The existence of the multipliers m_{ij} parallelly follows from the ideas of [7].

Let us define $B(\boldsymbol{u},\boldsymbol{v})\in V$ by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=(B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}), \quad \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V,$$

and set

$$\underline{\boldsymbol{F}}=\partial_t\boldsymbol{u}-\hat{\mu}\Delta\boldsymbol{u}+B(\boldsymbol{u},\boldsymbol{u})-\boldsymbol{F}_0, \tag{86}$$

where $\boldsymbol{F}_0=\boldsymbol{F}-b_s(\Theta-\Theta_0,\boldsymbol{u})$, as $b_s(\Theta-\Theta_0,\boldsymbol{u})\in L^2(\Omega)$ $\forall t$. Then $\underline{\boldsymbol{F}}\in L^2(I;V')$ and the variational inequality can be written in the form

$$(\underline{\boldsymbol{F}}(t),\boldsymbol{v})+\hat{g}j(\boldsymbol{v})-[(\underline{\boldsymbol{F}}(t),\boldsymbol{u}(t))+\hat{g}j(\boldsymbol{u}))]\geqslant 0 \quad \forall\boldsymbol{v}\in V. \tag{87}$$

Let us put $\boldsymbol{v}=\pm\lambda\boldsymbol{v}$, $\lambda>0$, then (87) leads to

$$\lambda[\pm(\underline{\boldsymbol{F}}(t),\boldsymbol{v})+\hat{g}j(\boldsymbol{v})]-[(\underline{\boldsymbol{F}}(t),\boldsymbol{u}(t))+\hat{g}j(\boldsymbol{u}(t))]\geqslant 0.$$

Hence, putting $\boldsymbol{v}=\boldsymbol{v}(t)$, where $t\to\boldsymbol{v}(t)\in L^2(I;V)$, then integrating over $t\in I$, we have

$$\lambda\left[\pm\int_I(\underline{\boldsymbol{F}}(t),\boldsymbol{v})\,\mathrm{d}t+J(\boldsymbol{v})\right]-\left[\int_I(\underline{\boldsymbol{F}}(t),\boldsymbol{u}(t))\,\mathrm{d}t+J(\boldsymbol{u}(t))\right]\geqslant 0 \quad \forall\boldsymbol{v}\in L^2(I;V), \quad \forall\lambda\geqslant 0. \tag{88}$$

Hence,

$$\left|\int_I (\underline{\boldsymbol{F}}(t),\boldsymbol{v})\,\mathrm{d}t\right| \leqslant J(\boldsymbol{v}) \quad \forall \boldsymbol{v}\in L^2(I;V), \tag{89}$$

and

$$\int_I (\underline{\boldsymbol{F}}(t),\boldsymbol{u}(t))\,\mathrm{d}t + J(\boldsymbol{u}(t)) = 0. \tag{90}$$

From (88) it follows that $\int_I (\underline{\boldsymbol{F}}(t),\boldsymbol{u}(t))\,\mathrm{d}t + J(\boldsymbol{u}(t)) \leqslant 0$ and since, due to (89), $\int_I [(\underline{\boldsymbol{F}}(t),\boldsymbol{u}(t))\,\mathrm{d}t + J(\boldsymbol{u}(t))\geqslant 0$, then (90) is valid.

Let us introduce the space

$$W_0 = \{w \mid w = \{w_{ij}\},\ w_{ij} = w_{ji},\ w_{ij} \in L^1(\Omega\times I)\},$$

with the norm

$$\|w\|_{W_0} = \int_{\Omega\times I} (w_{ij},w_{ij})^{1/2}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Let $P:\boldsymbol{v}\to D_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ be the mapping of $L^2(I;V)\to W_0$. Then (89) is equivalent to

$$\left|\int_I (\underline{\boldsymbol{F}}(t),\boldsymbol{v})\,\mathrm{d}t\right| \leqslant \hat{g}2^{1/2}\|P\boldsymbol{v}\|_{W_0}. \tag{91}$$

Applying the Hahn–Banach theorem,[9] then

$$m \in W_0' = \{w \mid w = \{w_{ij}\},\ w_{ij} = w_{ji},\ w_{ij} \in L^\infty(\Omega\times I)\} \quad \text{exists}$$

such that

$$\int_I (\underline{\boldsymbol{F}}(t),\boldsymbol{v})\,\mathrm{d}t = -\hat{g}2^{\frac{1}{2}} \int_{\Omega\times I} m_{ij} D_{ij}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{92}$$

and such that

$$\|m\|_{W_0'} \leqslant 1. \tag{93}$$

It is evident that we can take $\{m_{ij}\}$ in W_0' such that $m_{kk}=0$ as $D_{kk}=0$, W_0' is dual to W_0.

Eq. (93) is equivalent to $m_{ij}m_{ij}\leqslant 1$ a.e. in $\Omega\times I$. On virtue of definition of $\underline{\boldsymbol{F}}$, (92) is equivalent to

$$\partial_j \tau_{ij} + f_i = \rho\partial_t v_i, \quad \text{with } \tau_{ij} \text{ defined by (A3)(iv).}$$

Using (92) then (90) is equivalent to

$$\int_{\Omega\times I} m_{ij} D_{ij}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{\Omega\times I} (D_{ij}(\boldsymbol{u})D_{ij}(\boldsymbol{u}))^{1/2}\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

[9] *Hahn–Banach theorem*: Let φ be a continuous linear functional defined on a linear subset M of a normed linear space X. Then, there exists a continuous functional Φ, defined on X, such that $\Phi(u)=\varphi(u)$ for $u\in M$ and $\|\Phi\| = \|\varphi\|$.

Hence, since due to (A3)(ii) $m_{ij}m_{ij} \leqslant 1$ a.e. in $\Omega \times I$, then

$$m_{ij}D_{ij}(\boldsymbol{u}) = (D_{ij}(\boldsymbol{u})D_{ij}(\boldsymbol{u}))^{1/2} \quad \text{a.e. in } \Omega \times I.$$

Moreover, (90) is equivalent to

$$\partial_t u_i + \partial_j u_i u_j = \partial_j \tau_{ij} + f_i,$$

which completes this part of the proof.

Summing all the above obtained results we proved the existence of the weak solution of the problem investigated. □

Remark. In reality, the mushy zone is created not only by liquid and solid phases, but instead of the pure liquid microscopical elements of strongly viscoplastic rocks exist, but near to the liquid phase and having consistency of alloy of both phases. To determine the boundary between this phase and the solid visco-plastic phase is very difficult and can be determined by the determination of the threshold of plasticity $\hat{g}$, determining it by the voluminic fractions of both phases, the visco-plastic and the strongly visco-plastic with a low threshold of plasticity, so that $f_S + f_L = 1$. The dependence of the pressure and the temperature plays a fundamental role as shown in the phase diagrams. Therefore, the future problems must be studied with physical coefficients depending on the temperature and the pressure. Solving the problem $(\mathscr{P}_{\text{centh}})_v$ numerically, then we find the pressure p directly (as a result of the Uzawa algorithm) and then put it into the physical parameters of rocks. The temperature will be obtained as a solution of the thermal part of the problem. By such a way we can study directly influences of temperatures and pressures in anomalous zones of the Earth as well as in anomalous conditions of technologic problems.

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